

Risk-sensitive control for multi-class many server queues in the moderate deviation regime

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Abstract

A $G/M/N$ queue is considered in a moderate deviation heavy traffic regime. The rate functional for the customers-in-system process is obtained for single class model. A risk-sensitive type control problem is considered for multi-class $G/M/N$ model under moderate deviation scaling and shown that the optimal control problem is related to a differential game problem.

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1 Introduction

Studying scaling limit is an established tradition in queuing theory. These include heavy traffic approximation that depends on central limit theorem (CLT) and large deviation (LD) approximations. Another interesting scaling, considered in queuing network, is moderate deviation (MD) scaling which includes an intermediate scaling of CLT and LD. MD scaling are consider when the queuing network is critically loaded. Therefore MD can seen as LD analogue for heavy traffic set up. Also some control problems in MD regime have interesting characteristic that also appears in the asymptotic regime alluded to LD and heavy traffic approximations [1].

There have been several works on MD scaling without dynamic control aspect. LD and MD for renewal processes are proved in [17]. Later in [16], Puhalskii obtains MD principle for queue length and waiting-time processes for single class single server network. Majewski [15] considers feedforward multi-class network with priority and obtains MD asymptotics for waiting time, idle time, queue length, departure and sojourn time processes. We refer to [19], [10], for various interesting aspects of MD regime. A dynamic control problem for multi-class

$G/G/1$ queue in MD regime is considered in [1] where the authors point out some interesting features of the problem similar to other asymptotic regimes.

So far MD asymptotics have not been considered in many server queuing network. In this article we introduce MD principle for the customers-in-system process in a many server network. We consider a single class $G/M/N$ queuing network where the arrival is given by a general renewal process and the service requirements are exponential. We show that the rate functional for customers-in-system process in MD regime changes depending on the growth rate N compare to arrival rate λ^n . It is shown that if $N = o(\lambda^n)$ then the rate functional in MD regime for customers-in-system process is governed by Skorohod map. But if $\frac{N}{\lambda^n} \not\rightarrow 0$ as $n \rightarrow \infty$, the governing dynamics for the rate functional are not reflection maps. It is worthwhile to mention that this problem can be seen as MD analogue of the scaling considered by Halfin and Whitt for $G/M/N$ queuing network in [12]. One may wish to consider the MD analysis for $G/G/N$ network but the problem is harder as one needs to consider an infinite dimensional set up for the problem.

We also consider a risk-sensitive type control problem for multi-class $G/M/N$ network when $N = o(\lambda^n)$. We consider \mathbf{I} different customer classes arriving to a parallel server system following \mathbf{I} independent renewal processes. Service time distributions are exponential with class dependent parameters. Each customer is served by one of the servers and servers are not allowed to serve more than one customers at the same time. The problem is to control $B^n = (B_1^n, \dots, B_I^n)$ where B_i^n denotes the number of class- i customers in service, so that the cost is minimized. Denoting by X_i^n , the number of class- i jobs in the n -th system, the scaled version is given by $\tilde{X}_i^n = \frac{X_i^n - \rho_i N}{b_n \sqrt{n}}$ where ρ_i denotes the limiting traffic intensity for class- i and $\lim b_n = \infty$, $\lim \frac{b_n}{\sqrt{n}} = 0$. The cost is given by

$$\frac{1}{b_n^2} \mathbb{E}[e^{b_n^2 (\int_0^T h(\tilde{X}^n(s)) ds + g(\tilde{X}^n(T)))}],$$

where $T > 0$, and h, g are given nonnegative functions. This risk-sensitive type of cost has been studied in literature for its own importance (see [2], [3], [18]). One of the important aspect of the exponential cost is that it penalizes large quantities heavily. This is one of the reason for considering exponential cost attached to the queue length or customers-in-system processes. Another interesting aspect of working in MD regime is that the limiting differential game (DG) is solvable [1].

It is also interesting to compare the control problem above with the existing similar control problems ([1], [3]). In [3], the authors consider a similar problem (with bounded h) for multi-class $M/M/N$ network in LD regime. The convergence of the value function, corresponding to above optimal control problem, is proved using martingale method. In [1], a similar problem is considered for multi-class $G/G/1$ network and the convergence result is obtained by constructing a particular policy. In both the problems, the servers are allowed to serve more than one customers simultaneously. First of all, our proof technique here does not use any PDE analysis like [3]. Also in our above described model, we do not allow processor sharing. So the set of controls considered in this paper is smaller than those that are considered in earlier works. The proof of the convergence of the value function for the optimal control problem is divided into two parts. We first prove the lower bound estimate following similar technique as [1]. The proof for the upper bound is based on the construction of a particular policy such

that the lower bound is asymptotically attained. The construction of this policy is complicated than that appear in [1] and can be used to improve the control set used in [1]. We also obtain a simple control when the cost functions are linear and $N = o(b_n\sqrt{n})$. [6, 4] deal with a multi-class $G/M/N$ network under diffusion scaling where $N \approx \sqrt{n}$. Our problem can also be thought of as a generalization to these works in risk-sensitive set up. Let us also mention a related work [5] where a multi-class scheduling problem is considered under diffusion scaling.

To summarize the main contribution of the paper, we have (a) introduced the moderate deviation scaling for many server queues in heavy traffic regime, (b) shown the convergence of value function for optimal control problem to a value function of DG, (c) considered a smaller class of *admissible* control which can also be used to improve the results in [1], (d) given a simple policy when the cost functions are linear and $N = o(b_n\sqrt{n})$.

Notations: For a positive integer k and $a, b \in \mathbb{R}^k$, $a \cdot b$ denotes the usual scalar product, while $\|\cdot\|$ denotes Euclidean norm. For $a \in \mathbb{R}_+$, $[a]$ denote the largest integer less than or equal to a . Given $a, b \in \mathbb{R}$, the maximum (minimum) is denoted by $a \vee b$ ($a \wedge b$). We use a^+ (a^-) for $a \vee 0$ ($-a \vee 0$). Given two sequences $\{a_n\}$, $\{b_n\}$, $a_n = o(b_n)$ means $\limsup \frac{a_n}{b_n} = 0$. By \mathbb{R}_+^k we denote the nonnegative orthant of the Euclidean space \mathbb{R}^k . For $T > 0$ and a function $f : [0, T] \rightarrow \mathbb{R}^k$, let $\|f\|_t^* = \sup_{s \in [0, t]} \|f(s)\|$, $t \in [0, T]$. When $k = 1$, we write $|f|_t^*$ for $\|f\|_t^*$ and $\|f\|_T^*$ for $\|f\|_T^*$. $\mathbf{e}(\cdot)$ is used to denote the identity function on \mathbb{R} . For $T \leq \infty$, denote by $C([0, T], \mathbb{R}^k)$ and $D([0, T], \mathbb{R}^k)$ the spaces of continuous functions $[0, T] \rightarrow \mathbb{R}^k$ and, respectively, functions that are right-continuous with finite left limits (RCLL). For fix $T > 0$, endow the space $D([0, T], \mathbb{R}^k)$ with the Skorohod-Prohorov-Lindvall metric or J_1 metric, defined as

$$\mathbf{d}(\varphi, \varphi') = \inf_{f \in \mathcal{T}} \left(\|f\|^\circ \vee \sup_{[0, T]} \|\varphi(t) - \varphi'(f(t))\| \right), \quad \varphi, \varphi' \in D([0, T], \mathbb{R}^k)$$

where \mathcal{T} is the set of strictly increasing, continuous functions from $[0, T]$ onto itself, and

$$\|f\|^\circ = \sup_{0 \leq s < t \leq T} \left| \log \frac{f(t) - f(s)}{t - s} \right|.$$

As is well known [7], $D([0, T], \mathbb{R}^k)$ is a Polish space under the induced topology. Throughout this article, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All the stochastic processes introduced in this paper are defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

The paper is organized as follows. The next paragraph introduces some preliminaries that will be used in this paper. Section 2 introduces moderate deviation principle for a single class $G/M/N$ queue. Section 3 is devoted to the study of the multi-class $G/M/N$ queues and the dynamic control problem. Section 3.1 introduces the associated dynamic games and states the main results. The proof of the main results are given in Section 3.2. Finally, in Section 3.3 we prescribe a simple control which is asymptotically optimal when the cost functions are linear.

Preliminaries: Now we state the definition and properties of large deviation principle (LDP) and Skorohod problem that will be used in this paper. Given a metric space \mathcal{S} , a function \mathbb{I} , defined on \mathcal{S} , is said to be a *rate function* if the set $\{x \in \mathcal{S} : \mathbb{I}(x) \leq a\}$ is compact for all $a \geq 0$, and there is a sequence $\{\mathbb{P}_n\}_{n \geq 1}$ of probability measure on the Borel σ -field of \mathcal{S} (or sequence of random variable $\{X^n\}$ with law \mathbb{P}_n) satisfying large deviation principle with

parameter $a_n \rightarrow \infty$ and rate function \mathbb{I} i.e.,

$$\limsup \frac{1}{a_n} \log \mathbb{P}_n(F) \leq - \inf_{x \in F} \mathbb{I}(x),$$

for all closed set $F \subset \mathcal{S}$, and

$$\limsup \frac{1}{a_n} \log \mathbb{P}_n(F) \geq - \inf_{x \in G} \mathbb{I}(x),$$

for all closed set $G \subset \mathcal{S}$.

On standard way to get new LDP's from an existing one is through *contraction mapping principle* which states that if $\{X^n\}$ satisfies LDP with rate function \mathbb{I} and f is a continuous function on \mathcal{S} , then $f(X^n)$ satisfies LDP with rate function

$$\mathbb{I}_f(y) = \inf_{x: y=f(x)} \mathbb{I}(x). \quad (1.1)$$

There are several extension to this contraction mapping principle. We refer to [11] for a survey on contraction mapping principles. In this article, we use an *extended contraction mapping principle* which states that if $\{X^n\}$ obeys LDP with rate function \mathbb{I} , $\{f^n\}$ is a sequence of measurable functions, and if there is a measurable function f , continuous when restricted to the set $\{x : \mathbb{I}(x) \leq a\}$, $a \geq 0$, and $f^n(x^n) \rightarrow f(x)$ as $n \rightarrow \infty$ whenever $x^n \rightarrow x$ and $\mathbb{I}(x) < \infty$, then $\{f^n(X^n)\}_{n \geq 1}$ obeys LDP with rate function given by (1.1).

Our goal in this paper is to study asymptotics of certain value functions and to show that they lead to the value function of certain differential game problem. This differential game problem is solvable. In order to define the solution to the game we need do define *Skorohod problem*.

Definition 1.1 Let $\psi \in D([0, \infty), \mathbb{R})$ with $\psi(0) \in \mathbb{R}_+$ be given. Then (ϕ^1, ϕ^2) solves the Skorohod problem for the data ψ if $\psi(0) = \phi^1(0)$, and for all $t \in [0, \infty)$

1. $\phi^1(t) = \psi(t) + \phi^2(t)$,
2. $\phi^1(t) \in \mathbb{R}_+$,
3. ϕ^2 is nondecreasing,
4. $\int_0^\infty \phi^1(s) d\phi^2(s) = 0$.

It is know that the above problem has a unique solution ([13], [8]). Define $\Gamma(\psi) = \phi^1$. Γ is referred to as *Skorohod map*. In fact, Γ has an explicit form given by

$$\Gamma(\psi)(t) = \psi(t) + \sup_{0 \leq s \leq t} (\psi(s))^-.$$

It is easy to see that Γ satisfies Lipschitz property i.e.,

$$|\Gamma(\psi^1) - \Gamma(\psi^2)|_T^* \leq 2|\psi^1 - \psi^2|_T^*,$$

for $\psi^i \in D([0, \infty), \mathbb{R})$, $\psi^i(0) \in \mathbb{R}_+$, $i = 1, 2$.

2 Moderate deviations for many server queues

In this section, we introduce a single class $G/M/N$ model. We consider a parallel server system with single customer class and a pool of identical servers. Let λ^n be given parameter where $\frac{1}{\lambda^n}$ represents the mean of the inter-arrival times of customers in the n -th system. Let $\{IA(l) : l \in \mathbb{N}\}$ be a given sequence of i.i.d. of positive random variables with mean $\mathbb{E}[IA(1)] = 1$ and variance $\text{Var}(IA(1)) = \sigma_{i,IA}^2$. Assuming $\sum_1^0 = 0$, the number of arrivals of customers up to time t , for the n -th system, is given by

$$A^n(t) = \sup \left\{ l \geq 0 : \sum_{k=1}^l \frac{IA(k)}{\lambda^n} \leq t \right\}, \quad t \geq 0.$$

$N^n \in \mathbb{N}$ denotes the number of servers in the n -th system. Service time distributions are exponential. Let μ^n be the rate at which customers are served in the n -th system. We also consider the *moderate deviation rate* parameters $\{b_n\}$ with the property that $\lim b_n = \infty$ while $\lim \frac{b_n}{\sqrt{n}} = 0$. We assume that as $n \rightarrow \infty$,

$$\frac{\lambda^n}{n} \rightarrow \lambda > 0, \quad \frac{N^n \mu^n}{n} \rightarrow \mu > 0, \quad \frac{\sqrt{n}}{b_n} \left(\frac{\lambda^n}{n} - \frac{N^n \mu^n}{n} \right) \rightarrow r \in (-\infty, \infty). \quad (2.1)$$

It is easy to see that under (2.1), $\lambda = \mu$ i.e., the system is critically loaded. A similar condition in [16] is referred to as *near-heavy-traffic* condition.

Let X^n denote the number of customers in the system. Let $S(\cdot)$ be a standard Poisson process independent of the arrival process. The number of service completions of jobs by time t is given by

$$D^n(t) = S(\mu^n \int_0^t Z^n(s) ds), \quad (2.2)$$

where Z^n denote the number of customers in service. Hence we have

$$X^n(t) = X^n(0) + A^n(t) - D^n(t). \quad (2.3)$$

The system is assumed to work under non-idling policy i.e., $Z^n = X^n \wedge N^n$. Next we define the scaled process as follows

$$\tilde{A}^n(t) = \frac{1}{b_n \sqrt{n}} (A^n(t) - \lambda^n t), \quad \tilde{S}_\mu^n(t) = \frac{1}{b_n \sqrt{n}} (S(N^n \mu^n t) - N^n \mu^n t), \quad \tilde{X}^n(t) = \frac{1}{b_n \sqrt{n}} (X^n(t) - N^n). \quad (2.4)$$

It is easy to see from (2.3) that

$$\begin{aligned} \tilde{X}^n(t) &= \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n \left(\frac{1}{N^n} T^n(t) \right) + \frac{N^n \mu^n}{n} \frac{\sqrt{n}}{b_n} \left(t - \frac{1}{N^n} T^n(t) \right) \\ &= \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n \left(\frac{1}{N^n} T^n(t) \right) + \frac{N^n \mu^n}{n} \frac{n}{N^n} \int_0^t (\tilde{X}^n(s))^- ds, \end{aligned} \quad (2.5)$$

where $y^n = \frac{\sqrt{n}}{b_n} \left(\frac{\lambda^n}{n} - \frac{N^n \mu^n}{n} \right)$, $T^n(t) = \int_0^t Z^n(s) ds$. We fix $T > 0$ and assume:

Condition 2.1 *The process $(\tilde{A}^n, \tilde{S}_\mu^n)$ satisfies large deviation principle (LDP) in $D([0, T], \mathbb{R}^2)$ with parameter b_n^2 and rate function \mathbb{I} taking value ∞ on discontinuous paths.*

Remark 2.1 *Because of independence, it is enough if the processes A^n, S_μ^n satisfy LDP individually. In fact, one can impose some sufficient conditions on the inter-arrival processes so that Condition 2.1 holds (see Remark 3.2 and 3.3 below).*

We also assume that the initial condition is deterministic and

$$\tilde{X}^n(0) \rightarrow x \in \mathbb{R}, \quad \text{as } n \rightarrow \infty.$$

We are interested to find the rate function for \tilde{X}^n . We subdivide the problem in two theorems.

Theorem 2.1 *Assume Condition 2.1 holds and $N^n = n$. Then $\{X^n\}_{n \geq 1}$ defined in (2.4) satisfies LDP in $D([0, T], \mathbb{R})$ with parameter b_n^2 and rate function \mathbb{I}_X given by*

$$\mathbb{I}_X(\psi) = \inf_{\psi=G(\tilde{\psi}^1, \tilde{\psi}^2)} \mathbb{I}(\tilde{\psi}^1, \tilde{\psi}^2),$$

where $G(\tilde{\psi}^1, \tilde{\psi}^2)$ denotes the solution to the equation

$$\psi = x + rt + \tilde{\psi}^1(t) - \tilde{\psi}^2(t) + \mu \int_0^t (\psi(s))^- ds. \quad (2.6)$$

Proof: From (2.5), we have

$$\tilde{X}^n(t) = \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n\left(\frac{1}{N^n} T^n(t)\right) + \frac{N^n \mu^n}{n} \int_0^t (\tilde{X}^n(s))^- ds. \quad (2.7)$$

Now given any tuple $(\tilde{x}, \tilde{y}, \kappa, \tilde{\psi}^1, \tilde{\psi}^2) \in \mathbb{R}^3 \times D([0, T], \mathbb{R}^2)$ it is easy to see that there exists a unique $\xi \in D([0, T], \mathbb{R})$ satisfying the following:

$$\xi(t) = \tilde{x} + \tilde{y}t + \tilde{\psi}^1(t) - \tilde{\psi}^2(t) + \kappa \int_0^t (\xi(s))^- ds, \quad (2.8)$$

$$|\xi|_T^* \leq e^{\kappa T} \left(|\tilde{x} + \tilde{y}t + \tilde{\psi}^1(t) + \tilde{\psi}^2(t)|_T^* \right). \quad (2.9)$$

Since $(\tilde{A}^n, \tilde{S}_\mu^n)$ satisfies LDP with rate function \mathbb{I} and $\{\mathbb{I} \leq a\}, a \geq 0$, is compact, we have

$$\lim_{\alpha \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}^{\frac{1}{b_n^2}}(|\tilde{A}^n|_T^* + |\tilde{S}_\mu^n|_T^* \geq \alpha) = 0. \quad (2.10)$$

Now for any $\delta > 0$,

$$\mathbb{P}\left(\left|t - \frac{1}{N^n} T^n(t)\right|_T^* > \delta\right) \leq \mathbb{P}\left(e^{\frac{N^n \mu_i^n}{n} T} |\tilde{X}^n(0) + y^n t + \tilde{A}^n - \tilde{S}_{\mu_i}^n\left(\frac{1}{N^n} T^n(t)\right)|_T^* \geq \frac{\sqrt{n}}{b_n} \delta\right),$$

where we have used (2.9). Therefore using (2.1) and (2.10), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}^{\frac{1}{b_n^2}}\left(\left|t - \frac{1}{N^n} T^n(t)\right|_T^* > \delta\right) = 0. \quad (2.11)$$

Hence the sequence $\{\frac{1}{N^n}T^n\}$ converges super-exponentially in probability at rate $\frac{1}{b_n^2}$ to $\mathbf{e}(\cdot)$ in $D([0, T], \mathbb{R})$ where $\mathbf{e}(t) = t$. Therefore $(\tilde{A}^n, \tilde{S}_\mu^n \circ (\frac{1}{N^n}T^n))$ satisfies LDP with rate function \mathbb{I} in $D([0, T], \mathbb{R}^2)$ ([17], Lemma 4.3). Denote ξ by $G(\tilde{x}, \tilde{y}, \tilde{\psi}^1, \tilde{\psi}^2, \kappa)$ where ξ satisfies (2.8). Let $(\tilde{\psi}_n^1, \tilde{\psi}_n^2) \rightarrow (\tilde{\psi}^1, \tilde{\psi}^2)$ for some continuous path $(\tilde{\psi}^1, \tilde{\psi}^2)$. Let $\xi^n = G(\tilde{X}^n(0), y^n, \tilde{\psi}_n^1, \tilde{\psi}_n^2, \kappa^n)$, $\kappa^n = \frac{N^n \mu^n}{n}$, and $\xi = G(x, r, \tilde{\psi}^1, \tilde{\psi}^2, \mu)$. Then it is easy to see that $|\xi^n - \xi|_T^* \rightarrow 0$ as $n \rightarrow \infty$. Therefore extended contraction mapping principle yields that X^n satisfies LDP with parameter b_n^2 and rate function

$$\mathbb{I}_X(\psi) = \inf_{\psi=G(\tilde{\psi}^1, \tilde{\psi}^2)} \mathbb{I}(\tilde{\psi}^1, \tilde{\psi}^2),$$

where $G(\tilde{\psi}^1, \tilde{\psi}^2)$ denotes the solution to (2.6). \square

Theorem 2.2 *Assume Condition 2.1 holds. Let $N^n = o(n)$ and $x \in \mathbb{R}_+$. Then $\{X^n\}_{n \geq 1}$ defined in (2.4) satisfies LDP in $D([0, T], \mathbb{R})$ with parameter b_n^2 and rate function $\bar{\mathbb{I}}_X$ given by*

$$\bar{\mathbb{I}}_X(\psi) = \inf_{\psi=\Gamma(x+y\mathbf{e}+\tilde{\psi}^1-\tilde{\psi}^2)} \mathbb{I}(\tilde{\psi}^1, \tilde{\psi}^2),$$

where $\Gamma(\cdot)$ denotes the Skorohod map.

Proof: From (2.5), we have

$$\tilde{X}^n(t) = \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n(\frac{1}{N^n}T^n(t)) + \frac{N^n \mu^n}{n} \frac{n}{N^n} \int_0^t (\tilde{X}^n(s))^- ds. \quad (2.12)$$

By our assumption on N^n , we have $\frac{n}{N^n} \rightarrow \infty$ as $n \rightarrow \infty$. Given $\delta > 0$, we define the δ -oscillation function $osc_\delta : D([0, T], \mathbb{R}) \rightarrow \mathbb{R}_+$ as follows:

$$osc_\delta(\psi) = \sup\{|\psi(t) - \psi(s)| : |t - s| \leq \delta\}.$$

Applying the extended contraction mapping principle we see that $(osc_\delta(\tilde{A}^n), osc_\delta(\tilde{S}_\mu^n))$ satisfies LDP in \mathbb{R}_+^2 with rate function \mathbb{I}_0 given by

$$\mathbb{I}_0(x, y) = \begin{cases} 0 & \text{for } (x, y) = (0, 0), \\ \infty & \text{otherwise.} \end{cases}$$

Therefore, given any $\delta_1 > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}^{\frac{1}{b_n^2}}(osc_\delta(\tilde{A}^n) + osc_\delta(\tilde{S}_\mu^n) \geq \delta_1) = 0. \quad (2.13)$$

Now choose $\varepsilon > 0$. We claim that

$$\limsup_{n \rightarrow \infty} \mathbb{P}^{\frac{1}{b_n^2}}(|(\tilde{X}^n)^-|_T^* \geq 3\varepsilon) = 0. \quad (2.14)$$

Define $\Omega^n := \{(|(\tilde{X}^n)^-|_T^* \geq 3\varepsilon)\}$. Choose n large enough so that $\tilde{X}^n(0) \geq -\frac{\varepsilon}{2}$. For each $\omega \in \Omega^n$, we will have random times $0 \leq \sigma_1^n < \sigma_2^n \leq T$ such that $\tilde{X}^n(\sigma_1^n) > -\frac{3\varepsilon}{2}$, $\tilde{X}^n(\sigma_2^n) \leq -\frac{5\varepsilon}{2}$, and

$\tilde{X}^n(s) \leq -\varepsilon$ on $[\sigma_1^n, \sigma_2^n]$. This is possible to do as the jump size of \tilde{X}^n is $\frac{1}{b_n\sqrt{n}}$. Hence from (2.12), we have

$$-\varepsilon \geq y^n(\sigma_2^n - \sigma_1^n) - \text{osc}_{\sigma_2^n - \sigma_1^n}(\tilde{A}^n) - \text{osc}_{\sigma_2^n - \sigma_1^n}(\tilde{S}_\mu^n) + \frac{N^n \mu^n}{n} \frac{n}{N^n} \varepsilon (\sigma_2^n - \sigma_1^n). \quad (2.15)$$

Now if $(\sigma_2^n - \sigma_1^n) \geq \delta_1$ for some fix $\delta_1 > 0$ then (2.15) implies that $2(|y^n|T + |\tilde{A}^n|_T^* + |\tilde{S}_\mu^n|_T^*) \geq \frac{N^n \mu^n}{n} \frac{n}{N^n} \varepsilon \delta_1$. If $(\sigma_2^n - \sigma_1^n) < \delta_1$ then $-|y^n \delta_1| + \text{osc}_{\delta_1}(\tilde{A}^n) + \text{osc}_{\delta_1}(\tilde{S}_\mu^n) \geq \varepsilon$. Therefore if we choose $\delta_1 > 0$ so that $|y^n \delta_1| < \frac{\varepsilon}{2}$ for all n large, then

$$\mathbb{P}(|(\tilde{X}^n)^-|_T^* \geq 3\varepsilon) \leq \mathbb{P}(|\tilde{A}^n|_T^* + |\tilde{S}_\mu^n|_T^* \geq \kappa(\frac{n}{N^n})\varepsilon\delta_1) + \mathbb{P}(\text{osc}_{\delta_1}(\tilde{A}^n) + \text{osc}_{\delta_1}(\tilde{S}_\mu^n) \geq \varepsilon/2),$$

where $\kappa(\frac{n}{N^n}) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore using (2.10) and (2.13), the claim (2.14) follows. Now rewriting (2.12) as

$$(\tilde{X}^n(t))^+ = (\tilde{X}^n(t))^- + \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n(\frac{1}{N^n} T^n(t)) + \frac{N^n \mu^n}{n} \frac{n}{N^n} \int_0^t (\tilde{X}^n(s))^- ds, \quad (2.16)$$

we see that $(\tilde{X}^n(t))^+$ solves Skorohod problem for the data $(\tilde{X}^n(t))^- + \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n(\frac{1}{N^n} T^n(t))$. Hence using the Lipschitz property of the Skorohod map we have

$$|\frac{N^n \mu^n}{n} \frac{n}{N^n} \int_0^t (\tilde{X}^n(s))^- ds|_T^* \leq 2|(\tilde{X}^n(t))^- + \tilde{X}^n(0) + y^n t + \tilde{A}^n(t) - \tilde{S}_\mu^n(\frac{1}{N^n} T^n(t))|_T^*. \quad (2.17)$$

Since $\frac{\sqrt{n}}{b_n}(t - \frac{1}{N^n} T^n(t)) = \frac{n}{N^n} \int_0^t (\tilde{X}^n(s))^- ds$, applying (2.10), (2.14) and (2.17), we have

$$\limsup_{n \rightarrow \infty} \mathbb{P}^{\frac{1}{b_n^2}}(|t - \frac{1}{N^n} T^n(t)|_T^* > \delta) = 0.$$

Hence $(\tilde{A}^n, \tilde{S}_\mu^n \circ (\frac{1}{N^n} T^n))$ satisfies LDP with rate function \mathbb{I} ([17], Lemma 4.3). Now we consider a sequence $(\tilde{\psi}_n^1, \tilde{\psi}_n^2) \rightarrow (\tilde{\psi}^1, \tilde{\psi}^2)$ as $n \rightarrow \infty$ for some continuous path $(\tilde{\psi}^1, \tilde{\psi}^2) \in D([0, T], \mathbb{R}^2)$. Let ξ^n be the solution to (2.8) with the data $(\tilde{X}^n(0), y^n, \frac{N^n \mu^n}{n} \frac{n}{N^n}, \tilde{\psi}_n^1, \tilde{\psi}_n^2)$. Let $\xi = \Gamma(x + y\mathbf{e} + \tilde{\psi}^1 - \tilde{\psi}^2)$. To complete the proof it is enough to show that $|\xi^n - \xi|_T^* \rightarrow 0$ as $n \rightarrow \infty$. The proof will follow from the extended contraction mapping principle. Given $\varepsilon > 0$, we choose $\delta > 0$ such that $(\omega_\delta(\psi_n^1) + \omega_\delta(\psi_n^2)) < \frac{\varepsilon}{4}$ for all n large. Since $\sup_n(|\tilde{\psi}_n^1|_T^* + |\tilde{\psi}_n^2|_T^*) < \infty$, we can choose δ small enough to conclude that

$$|(\xi^n)^-|_T^* \leq 3\varepsilon,$$

for large n (using (2.15)). From (2.8), we note that $(\xi^n)^+$ satisfies Skorohod problem for the data $(\xi^n(\cdot))^- + \tilde{X}^n(0) + y^n \mathbf{e}(\cdot) + \tilde{\psi}_n^1(\cdot) - \tilde{\psi}_n^2(\cdot)$ and therefore Lipschitz property of the Skorohod map implies

$$\begin{aligned} |(\xi^n)^+ - \xi|_T^* &\leq 2|(\xi^n(\cdot))^- + \tilde{X}^n(0) + y^n \mathbf{e}(\cdot) + \tilde{\psi}_n^1(\cdot) - \tilde{\psi}_n^2(\cdot) - (x + r\mathbf{e}(\cdot) + \tilde{\psi}^1(\cdot) - \tilde{\psi}^2(\cdot))|_T^*, \\ &\leq 8\varepsilon, \end{aligned}$$

for all n large. Hence $|\xi^n - \xi|_T^* \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. \square

3 Control of multi-class $G/M/N$

In this section, we introduce a multi-class $G/M/N$ model and a related control problem. We consider a parallel server system with \mathbf{I} number of customer classes and a pool of identical servers. Let $\mathcal{I} = \{1, 2, \dots, \mathbf{I}\}$. Let $\lambda_i^n > 0, n \in \mathbb{N}, i \in \mathcal{I}$, be given parameter where $\frac{1}{\lambda_i^n}$ represents the mean of the inter-arrival time of class- i customers in the n -th system. Given are \mathbf{I} independent sequence $\{IA_i(l) : l \in \mathbb{N}\}_{i \in \mathcal{I}}$ of positive random variables with mean $\mathbb{E}[IA_i(1)] = 1$ and variance $\text{Var}(IA_i(1)) = \sigma_{i,IA}^2$. Assuming $\sum_1^0 = 0$, the number of arrivals of class- i customers up to time t , in the n -th system, is given by

$$A_i^n(t) = \sup \left\{ l \geq 0 : \sum_{k=1}^l \frac{IA_i(k)}{\lambda_i^n} \leq t \right\}, \quad t \geq 0.$$

$N^n \in \mathbb{N}$ denotes the number of servers in the n -th system. Service time distributions are exponential, with class dependent parameter. Let μ_i^n be the rate at which class- i customers are served in the n -th system. We also consider the *moderate deviation rate* parameters $\{b_n\}$ with the property that $\lim b_n = \infty$ while $\lim_{n \rightarrow \infty} [\frac{b_n}{\sqrt{n}} \vee \frac{n}{N^n}] = 0$. Note that $N^n = o(n)$. We assume that as $n \rightarrow \infty$,

- $\frac{\lambda_i^n}{n} \rightarrow \lambda_i > 0$ and $\frac{N^n \mu_i^n}{n} \rightarrow \mu_i > 0$,
- $\tilde{\lambda}_i^n := \frac{1}{b_n \sqrt{n}} (\lambda_i^n - n \lambda_i) \rightarrow \tilde{\lambda}_i \in (-\infty, \infty)$,
- $\tilde{\mu}_i^n := \frac{1}{b_n \sqrt{n}} (N^n \mu_i^n - n \mu_i) \rightarrow \tilde{\mu}_i \in (-\infty, \infty)$.

Hence the traffic intensity for class- i , namely $\frac{\lambda_i^n}{N^n \mu_i^n}$, has limit $\rho_i := \frac{\lambda_i}{\mu_i}$. The system is assumed to be critically loaded i.e., $\sum_{i=1}^{\mathbf{I}} \rho_i = 1$.

Let $B_i^n(t)$ be the number of servers working on class- i customers at time $t \geq 0$. Therefore $B^n = (B_1^n, \dots, B_{\mathbf{I}}^n)$ takes value in $\mathbb{N}^{\mathbf{I}}$. Let X_i^n, Q_i^n, I^n denote the number of class- i customers in the system, the queue length of class- i customers in the buffer and the number of servers that are idle, respectively. Hence we have

$$X_i^n = Q_i^n + B_i^n, \quad i \in \mathcal{I}, \quad (3.1)$$

$$N^n = I^n + \sum_{i \in \mathcal{I}} B_i^n. \quad (3.2)$$

We are given \mathbf{I} independent standard Poisson processes $S_i, i \in \mathcal{I}$. The number of service completions of class- i jobs by time t is given by

$$D_i^n(t) = S_i(\mu_i^n T_i^n(t)), \quad (3.3)$$

where

$$T_i^n = \int_0^t B_i^n(s) ds. \quad (3.4)$$

Hence we have

$$X_i^n(t) = X_i^n(0) + A_i^n(t) - D_i^n(t). \quad (3.5)$$

For simplicity, the initial condition $X^n = (X_i^n, \dots, X_I^n)$ is assumed to be deterministic. The processes A^n, X^n, Q^n, B^n will always be assumed to have RCLL sample paths. We will also assume that the processes $A_i^n, S_i, i \in \mathcal{I}$, are mutually independent.

The process B^n is regarded as control, that is determined based on the observation from the past (and present) events in the system. Fix $T > 0$. Given n , the process B^n is said to be an *admissible control* if its sample paths lie in $D([0, T], \mathbb{R}_+^{\mathbf{I}})$ and

- $B^n(t) \in \mathbb{N}^{\mathbf{I}}$ for all $t \geq 0$;
- For $i \in \mathcal{I}$ and $t \geq 0$,
$$B_i^n(t) \leq X_i^n, \quad \text{and} \quad \sum_{i \in \mathcal{I}} B_i^n(t) \leq N^n; \quad (3.6)$$

- It is adapted to the filtration

$$\sigma\{A_i^n(s), D_i^n(s), i \in \mathcal{I}, s \leq t\}.$$

Denote the class of all admissible controls B^n by \mathfrak{U}^n . We can see that under admissible control each server is allowed to serve a single customer at a time. We do not allow processor sharing.

Next we introduce the scaled processes. For $i \in \mathcal{I}$, let

$$\begin{aligned} \tilde{A}_i^n(t) &= \frac{1}{b_n \sqrt{n}}(A_i^n(t) - \lambda_i^n t), \quad \tilde{S}_{\mu_i}^n(t) = \frac{1}{b_n \sqrt{n}}(S_i^n(N^n \mu_i^n t) - N^n \mu_i^n t), \\ \tilde{X}_i^n(t) &= \frac{1}{b_n \sqrt{n}}(X_i^n(t) - \rho_i N^n). \end{aligned} \quad (3.7)$$

It is easy to check from (3.5) that

$$\tilde{X}_i^n(t) = \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n\left(\frac{1}{N^n} T_i^n(t)\right) + Z_i^n(t), \quad (3.8)$$

where we denote

$$Z_i^n(t) = \frac{N^n \mu_i^n}{n} \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} T_i^n(t)), \quad y_i^n = \tilde{\lambda}_i^n - \rho_i \tilde{\mu}_i^n. \quad (3.9)$$

Since $\sum_{i \in \mathcal{I}} B_i^n \leq N^n$ and $\sum_i \rho_i = 1$, we see that

$$\sum_i \frac{n}{N^n \mu_i^n} Z_i^n \quad \text{starts from zero and is nondecreasing,} \quad (3.10)$$

The initial condition $X_i^n(0)$ is assumed to satisfy the following:

$$\tilde{X}_i^n(0) \rightarrow x \in \mathbb{R}_+^{\mathbf{I}}, \quad \text{as } n \rightarrow \infty.$$

The scaled arrival processes \tilde{A}^n is assumed to satisfy a *moderate deviation principle*. Let us first define the rate functions. Let $\mathbb{I}_k, k = 1, 2$, be functions on $D([0, T], \mathbb{R}^{\mathbf{I}})$ defined as follows. For $\psi = (\psi_1, \dots, \psi_I) \in D([0, T], \mathbb{R}^{\mathbf{I}})$,

$$\mathbb{I}_1(\psi) = \begin{cases} \frac{1}{2} \sum_{i=1}^I \frac{1}{\lambda_i \sigma_{i,IA}^2} \int_0^T \dot{\psi}_i^2(s) ds & \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\mathbb{I}_2(\psi) = \begin{cases} \frac{1}{2} \sum_{i=1}^I \frac{1}{\mu_i} \int_0^T \dot{\psi}_i^2(s) ds & \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Condition 3.2 (Moderate deviation principle) *The sequence $\tilde{A}^n = (\tilde{A}_1^n, \dots, \tilde{A}_I^n)$, satisfies the LDP with parameters b_n^2 and rate function \mathbb{I}_1 in $D([0, T], \mathbb{R}^{\mathbf{I}})$; i.e.,*

- *For any closed set $F \subset D([0, T], \mathbb{R}^{\mathbf{I}})$*

$$\limsup \frac{1}{b_n^2} \log \mathbb{P}(\tilde{A}^n \in F) \leq - \inf_{\psi \in F} \mathbb{I}_1(\psi),$$

- *For any open set $G \subset D([0, T], \mathbb{R}^{\mathbf{I}})$*

$$\liminf \frac{1}{b_n^2} \log \mathbb{P}(\tilde{A}^n \in G) \geq - \inf_{\psi \in G} \mathbb{I}_1(\psi).$$

Remark 3.2 *It is shown in [17] that each one of the following statements is sufficient for Condition 3.2 to hold:*

- *There exist constants $a_0 > 0$, $\beta \in (0, 1]$ such that $E[e^{a_0(I A_i)^\beta}] < \infty$, $i \in \mathcal{I}$, and $b_n^{\beta-2} n^{\beta/2} \rightarrow \infty$;*
- *For some $\delta > 0$, $E[(I A_i)^{2+\delta}] < \infty$, $i \in \mathcal{I}$, and $b_n^{-2} \log n \rightarrow \infty$.*

Remark 3.3 *Since the inter-arrival time for a Poisson process is exponential, using Remark 3.2, we see that $\tilde{S}_\mu^n = (\tilde{S}_{\mu_1}^n, \dots, \tilde{S}_{\mu_I}^n)$, satisfies Large deviation principle in $D([0, T], \mathbb{R}^{\mathbf{I}})$ with parameter b_n^2 and rate function \mathbb{I}_2 . Therefore using the independence of the processes (see [14]) and extended contraction mapping principle we see that $(\tilde{A}^n, \tilde{S}_\mu^n)$ satisfies Large deviation principle in $D([0, T], \mathbb{R}^{2\mathbf{I}})$ with parameter b_n^2 and rate function $\mathbb{I}(\psi) = \mathbb{I}_1(\psi^1) + \mathbb{I}_2(\psi^2)$, $\psi = (\psi^1, \psi^2) \in D([0, T], \mathbb{R}^{2\mathbf{I}})$.*

To present our control problem, we consider nonnegative functions h and g from $\mathbb{R}^{\mathbf{I}}$ to \mathbb{R} which are nondecreasing with respect to the usual partial order on $\mathbb{R}^{\mathbf{I}}$. We assume that h, g have at most linear growth, i.e., there exist constants C_1, C_2 such that

$$g(x) + h(x) \leq C_1 \|x\| + C_2.$$

Given n , the cost associated with the initial condition $\tilde{X}^n(0)$ and control B^n is given by

$$J_X^n(\tilde{X}^n(0), B^n) = \frac{1}{b_n^2} \log \mathbb{E} \left[e^{b_n^2 [\int_0^T h(\tilde{X}^n(s)) ds + g(\tilde{X}^n(T))]} \right].$$

We are interested to analyze the value function

$$V_X^n(\tilde{X}^n(0)) = \inf_{B^n \in \mathcal{U}^n} J_X^n(\tilde{X}^n(0), B^n).$$

We now introduce another value function associated to the queue length. To do this, we define $\tilde{Q}_i^n = \frac{1}{b_n \sqrt{n}} Q_i^n$, $i \in \mathcal{I}$, and $\tilde{Q}^n = (\tilde{Q}_1^n, \dots, \tilde{Q}_{\mathbf{I}}^n)$. Let

$$J_Q^n(\tilde{Q}^n(0), B^n) = \frac{1}{b_n^2} \log \mathbb{E} \left[e^{b_n^2 \int_0^T h(\tilde{Q}^n(s)) ds + g(\tilde{Q}^n(T))} \right].$$

The associated value function is given by

$$V_Q^n(\tilde{Q}^n(0)) = \inf_{B^n \in \mathbb{B}^n} J_Q^n(\tilde{X}^n(0), B^n).$$

3.1 A differential game and main results

We next develop a differential game for the limiting behavior of the value functions defined above. This game problem has been studied in [1]. Let $\theta = (\frac{1}{\mu_1}, \dots, \frac{1}{\mu_{\mathbf{I}}})$ and $y = (y_1, \dots, y_{\mathbf{I}})$ where $y_i = \tilde{\lambda}_i - \rho_i \tilde{\mu}_i$. Denote $P = C_0([0, T], \mathbb{R}^{2\mathbf{I}})$ (the subset of $C([0, T], \mathbb{R}^{2\mathbf{I}})$ of functions with initial value 0) and

$$E = \{\zeta \in C([0, T], \mathbb{R}^{\mathbf{I}}) : \theta \cdot \zeta \text{ starts from zero and is nondecreasing}\}.$$

The topology on both the spaces are induced by uniform topology. Let R be a mapping from $D([0, T], \mathbb{R}^{\mathbf{I}})$ into itself defined by

$$R[\psi]_i(t) = \psi_i(\rho_i t), \quad t \in [0, T], \quad i \in \mathcal{I}.$$

Given $\psi = (\psi^1, \psi^2) \in P$ and $\zeta \in E$, we define the *dynamics associated with initial condition x and data ψ, ζ* as

$$\varphi_i(t) = x_i + y_i t + \psi_i^1(t) - R[\psi^2]_i(t) + \zeta_i(t), \quad i \in \mathcal{I}. \quad (3.11)$$

It is easy to see the analogy between the above equation and equation (3.8), and between the condition $\theta \cdot \zeta$ nondecreasing and property (3.10). The following condition will also be used,

$$\varphi_i(t) \geq 0, \quad t \geq 0, \quad i \in \mathcal{I}. \quad (3.12)$$

To define the game in the sense of Elliott and Kalton [9], we need the notion of strategies. A measurable mapping $\alpha : P \rightarrow E$ is called a *strategy for the minimizing player* if it satisfies the causality property. Namely, for every $\psi = (\psi^1, \psi^2), \tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in P$ and $t \in [0, T]$,

$$(\psi^1, R[\psi^2])(s) = (\tilde{\psi}^1, R[\tilde{\psi}^2])(s) \text{ for all } s \in [0, t] \quad \text{implies} \quad \alpha[\psi](s) = \alpha[\tilde{\psi}](s) \text{ for all } s \in [0, t]. \quad (3.13)$$

Given an initial condition x , a strategy α is said to be *admissible* if, for $\psi \in P$ and $\zeta = \alpha[\psi]$, the corresponding dynamics (3.11) satisfies the nonnegativity constraint (3.12). The set of all admissible strategies for the minimizing player is denoted by A . Given x and $(\psi, \zeta) \in P \times E$, we define the cost by

$$c(\psi, \zeta) = \int_0^T h(\varphi(t)) dt + g(\varphi(T)) - \mathbb{I}(\psi),$$

where φ is the corresponding dynamics given by (3.11) and \mathbb{I} is given in Remark 3.3. The value of the game is defined by

$$V(x) = \inf_{\alpha \in A_x} \sup_{\psi \in P} c(\psi, \alpha[\psi]).$$

One can also obtain a simpler, equivalent formulation of the above game (see Remark 2.2 in [1]).

3.1.1 Main results

Before we state our main results, let us introduce two conditions that will be used to prove the results. For $w \in \mathbb{R}_+$, define

$$h^*(w) = \inf\{h(x) : x \in \mathbb{R}_+^{\mathbf{I}}, \theta \cdot x = w\}, \quad g^*(w) = \inf\{g(x) : x \in \mathbb{R}_+^{\mathbf{I}}, \theta \cdot x = w\}.$$

We impose the following condition.

Condition 3.3 (Existence of a continuous minimizing curve) *There exists a continuous map $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^{\mathbf{I}}$ such that for all $w \in \mathbb{R}_+$,*

$$\theta \cdot f(w) = w, \quad h^*(w) = h(f(w)), \quad g^*(w) = g(f(w)).$$

We refer to [1] for the examples of h and g satisfying above condition. Similar condition is also used in [6], [4], where an analogous many-server model is treated in a diffusion regime.

Condition 3.4 (Exponential moments) *Denote $A_T(\psi^1) = \sum_{i=1}^{\mathbf{I}} \sup_{[0,T]} |\psi_i^1(t)|$. Then for any constant K ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E}[e^{b_n^2 K A_T(\tilde{A}^n)}] < \infty.$$

In view of Proposition 2.1 in [1], if there exists $a_0 > 0$ such that $\sup_{i \in \mathcal{I}} \mathbb{E}[e^{a_0 I A_i}] < \infty$ then Condition 3.4 holds.

Remark 3.4 *If Condition 3.4 holds, then it is easy to see that for any constant K ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{b_n^2} \log \mathbb{E}[e^{b_n^2 K (A_T(\tilde{A}^n) + A_T(\tilde{S}_\mu^n))}] < \infty.$$

Now we are ready to state our main results.

Theorem 3.3 *Let Conditions 3.2, 3.3 and 3.4 hold and $\lim_{n \rightarrow \infty} \frac{N^n}{b_n \sqrt{n}} = 0$. Then $\lim V_X^n(\tilde{X}_n(0)) = V(x)$.*

Theorem 3.4 *Let Conditions 3.2, 3.3 and 3.4 hold and $\lim_{n \rightarrow \infty} \frac{N^n}{b_n \sqrt{n}} = 0$. Then $\lim V_Q^n(\tilde{Q}^n(0)) = V(x)$.*

Theorem 3.5 *Let Conditions 3.2 and 3.3 hold. If h and g are bounded, then $\lim_{n \rightarrow \infty} V_X^n(\tilde{X}^n(0)) = V(x)$.*

3.2 Proof of Theorem 3.3, 3.4 and 3.5

3.2.1 Lower bound

Before we go in further details, let us mention a solution to the above game problem that was obtained in [1]. Recall the one-dimensional Skorohod map Γ from $D([0, T], \mathbb{R})$ into itself. Given $\psi = (\psi^1, \psi^2) \in P$, define

$$\hat{\psi}(t) = x + yt + \psi^1(t) - \psi^2(t), \quad t \in [0, T].$$

We define

$$\hat{\alpha}_\theta[\psi](t) = f(\hat{\varphi}_\theta[\psi](t)) - \hat{\psi}(t), \quad t \in [0, T], \quad (3.14)$$

where $\hat{\varphi}_\theta[\psi] = I[\theta \cdot \hat{\psi}]$. Let us define $\alpha_\theta[\psi^1, \psi^2] = \hat{\alpha}_\theta[\psi^1, R[\psi^2]]$. In [1], it is proved that α_θ is a minimizing strategy for the game i.e.,

$$V(x) = \sup_{\psi \in P} c(\psi, \alpha_\theta[\psi]). \quad (3.15)$$

For $\kappa > 0$, we define

$$D(\kappa) = \{\psi = (\psi^1, \psi^2) \in D([0, T], \mathbb{R}^{2\mathbf{I}}) : \|\psi^1\|_T^* + \|\psi^2\|_T^* \leq \kappa \text{ and } \bar{\psi}(0) \in \mathbb{R}_+^{\mathbf{I}}\}, \quad (3.16)$$

where

$$\bar{\psi}(t) = x + yt + \psi^1(t) - R[\psi^2](t), \quad t \in [0, T].$$

It is shown in ([1], (27)) that there exists constant γ_1, γ_2 such that

$$\|\hat{\alpha}_\theta[\psi]\|_t^* \leq \gamma_1(A_t(\psi^1) + A_t(\psi^2) + \gamma_2), \quad t \in [0, T], \quad (3.17)$$

for all $(\psi^1, \psi^2) \in D([0, T], \mathbb{R}^{2\mathbf{I}})$. Given a map $\varphi : [0, T] \rightarrow \mathbb{R}^k$ and a constant $\eta > 0$, we define the η -oscillation of φ as

$$\text{osc}_\eta(\varphi) = \sup\{\|\varphi(s) - \varphi(t)\| : |s - t| \leq \eta, s, t \in [0, T]\}.$$

Then for any given $\kappa, \varepsilon > 0$ there exists δ, η such that the followings hold: For any $\psi, \tilde{\psi} \in D(\kappa)$

$$\|\hat{\alpha}_\theta[\psi] - \hat{\alpha}_\theta[\tilde{\psi}]\|_T^* \leq \varepsilon \quad \text{if} \quad \|\psi^1 - \tilde{\psi}^1\|^* + \|\psi^2 - \tilde{\psi}^2\|^* \leq \delta, \quad (3.18)$$

and

$$\text{osc}_\eta(\hat{\alpha}_\theta[\psi]) \leq \varepsilon \quad \text{provided} \quad \text{osc}_\eta(\psi) \leq \delta. \quad (3.19)$$

Theorem 3.6 *Assume Conditions 3.2 and 3.3 hold. Then $\liminf V_X^n(\tilde{X}_n(0)) \geq V(x)$.*

Proof: The proof of the theorem follows from [1] except some suitable modifications. We add here some details for clarity and convenience of the readers. Fix $\tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in P$. Recall metric $\mathbf{d}(\cdot, \cdot)$ on $D([0, T], \mathbb{R}^{2\mathbf{I}})$ which induces the J_1 topology. Define, for $r > 0$,

$$\mathcal{A}_r = \{\psi \in D([0, T], \mathbb{R}^{2\mathbf{I}}) : \mathbf{d}(\psi, \tilde{\psi}) < r\}.$$

Since $\tilde{\psi}$ is continuous, for any $r_1 \in (0, 1)$ there exists $r, \eta > 0$ such that

$$\psi \in \mathcal{A}_r \quad \text{implies} \quad \|\psi - \tilde{\psi}\|^* < r_1, \quad \text{osc}_\eta(\psi) < r_1. \quad (3.20)$$

This can be done as for any $f \in \mathcal{Y}$ (see Notations),

$$\begin{aligned} \|\psi(t) - \tilde{\psi}(t)\| &\leq \|\psi(t) - \tilde{\psi}(f(t))\| + \|\tilde{\psi}(f(t)) - \tilde{\psi}(t)\|, \\ |f(t) - t|_T^* &\leq T(e^{\|f\|^\circ} - 1), \end{aligned}$$

and $\tilde{\psi}$ is uniformly continuous on $[0, T]$. Define $\theta^n = (\frac{n}{N^n \mu_1^n}, \frac{n}{N^n \mu_2^n}, \dots, \frac{n}{N^n \mu_I^n})$. Then $\theta^n \rightarrow \theta$ as $n \rightarrow \infty$. Now, given $0 < \varepsilon < 1$, choose a sequence of policies $\{B^n\} \subset \mathfrak{U}^n$ such that

$$V_X^n(\tilde{X}^n(0)) + \varepsilon > J_X(\tilde{X}^n(0), B^n) \text{ and } B^n \in \mathfrak{U}^n \text{ for all } n.$$

Recall

$$\tilde{X}_i^n(t) = \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n(\frac{1}{N^n} T_i^n(t)) + Z_i^n(t), \quad (3.21)$$

where

$$Z_i^n(t) = \frac{N^n \mu_i^n}{n} \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} T_i^n(t)), \quad T_i^n(t) = \int_0^t B_i^n(s) ds. \quad (3.22)$$

We claim that for all n large and $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$,

$$\sup_{i \in \mathcal{I}} |(\tilde{X}_i^n)^-|_T^* \leq (6 + c)r_1, \quad (3.23)$$

where $c = \sup_n \|y^n\|$. To prove the claim, let us take $i \in \mathcal{I}$ such that $|(\tilde{X}_i^n)^-|_T^* > (6 + c)r_1$ for infinitely many n . For large n , we have $\tilde{X}_i^n(0) > -r_1$. Hence we have times $\sigma_1^n < \sigma_2^n \leq T$ such that $\tilde{X}_i^n(\sigma_1^n) \geq -2r_1$, $\tilde{X}_i^n(\sigma_2^n) \leq -(c + 5)r_1$ and $\tilde{X}_i^n(s) \leq -r_1$ for all $s \in [\sigma_1^n, \sigma_2^n]$. Therefore $\frac{B_i^n(s) - \rho_i N^n}{b_n \sqrt{n}} \leq \frac{X_i^n(s) - \rho_i N^n}{b_n \sqrt{n}} \leq -r_1$ for all $s \in [\sigma_1^n, \sigma_2^n]$. Therefore using (3.21) we have

$$-(c + 3)r_1 \geq y_i^n(\sigma_2^n - \sigma_1^n) - 2\text{osc}_{\sigma_2^n - \sigma_1^n}(\tilde{A}^n, \tilde{S}_\mu^n) + \frac{N^n \mu_i^n}{n} \frac{n}{N^n} r_1(\sigma_2^n - \sigma_1^n). \quad (3.24)$$

Using (3.20) and the fact $\frac{n}{N^n} \rightarrow \infty$, we see that (3.24) leads to a contradiction for large n if $(\sigma_2^n - \sigma_1^n) \geq r_1$. Again if $(\sigma_2^n - \sigma_1^n) \leq r_1$, then (3.24) is contradicting to (3.20) as the right most term in (3.24) is non-negative. This proves the claim (3.23).

Given $G > 0$, define

$$\tau_n = \inf\{t \geq 0 : \theta^n \cdot Z^n(t) > G\} \wedge T \equiv \inf\left\{t \geq 0 : \frac{\sqrt{n}}{b_n} \left(t - \frac{1}{N^n} \sum_{i=1}^I T_i^n(t)\right) > G\right\} \wedge T.$$

It is possible to choose $\kappa_1 > 0$ such that for $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$ and $t > \tau_n$ (see (38) in [1]),

$$\theta^n \cdot \tilde{X}^n(t) \geq -\kappa_1 + G. \quad (3.25)$$

Next, let $\varphi : [0, T] \rightarrow \mathbb{R}^I$ be the dynamics corresponding to $(\tilde{\psi}, \zeta)$, where $\zeta = \alpha_\theta[\tilde{\psi}]$, namely

$$\varphi_i(t) = x_i + y_i t + \tilde{\psi}_i^1(t) - R[\tilde{\psi}^2]_i(t) + \zeta_i(t). \quad (3.26)$$

To this end, we note that given $\kappa > 0$, for large n (so that $\theta^n \leq 2\theta$)

$$\theta^n \cdot x \geq \kappa \Rightarrow \theta \cdot x \geq \frac{\kappa}{2} + \frac{\theta^n \cdot x^-}{2} - \theta \cdot x^-. \quad (3.27)$$

Then $\varphi(t) = f(\varphi_\theta[\tilde{\psi}](t))$ (3.14). Let $\bar{\omega}_h$ [$\bar{\omega}_g$] be the modulus of continuity of h [resp. g] over $\{x \in \mathbb{R}^I : x \cdot \theta \leq |\varphi_\theta[\tilde{\psi}]|_T^*, x_i \geq -(c + 6)r_1\}$. For $\kappa, \frac{\kappa}{2} - 2\sqrt{I} \sup_n \|\theta^n\|(c + 6) > |\varphi_\theta[\tilde{\psi}]|_T^*$, and large n we have

$$\begin{aligned} \inf\{h(x) : \theta^n \cdot x \geq \kappa, x_i \geq -(c + 6)r_1\} &\geq \inf\{h(x) : \theta \cdot x \geq |\varphi_\theta[\tilde{\psi}]|_T^*, x_i \geq -(c + 6)r_1\} \\ &\geq \inf\{h(x) : x \in R_+^I, \theta \cdot x \geq |\varphi_\theta[\tilde{\psi}]|_T^*\} - \bar{\omega}_h(\sqrt{I}(c + 6)r_1) \\ &\geq |h(\varphi)|_T^* - \bar{\omega}_h(\sqrt{I}(c + 5)r_1), \end{aligned}$$

where for the first inequality we use (3.27) and for the last inequality we use the monotonicity of h . Therefore we can find $G > 0$ such that for all large n , $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$,

$$h(\tilde{X}^n(t)) \geq |h(\varphi)|^* - \delta(r_1) \text{ and } g(\tilde{X}_n(t)) \geq g(\varphi(T)) - \delta(r_1), \quad (3.28)$$

on $\{t > \tau_n\}$ where $\delta(r_1) \rightarrow 0$ as $r_1 \rightarrow 0$, and for $t \leq \tau_n$

$$\|Z_n(t)\| \leq \kappa_2. \quad (3.29)$$

for some constant κ_2 . Hence using (3.21), (3.20) and (3.29), we obtain a constant κ_3 such that for $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$ and all n large

$$\sup_{t \in [0, \tau_n]} \|\tilde{X}_i^n(t)\| \leq \kappa_3.$$

Consider the stochastic processes $Y^n, \tilde{Y}^n, \tilde{Z}^n$, with values in $\mathbb{R}^{\mathbf{I}}$ such that $(\tilde{X}_i^n)^+(t) = x_i + y_i t + Y_i^n(t) - \tilde{Y}_i^n(t) + \tilde{Z}_i^n(t)$ on $[0, \tau_n]$ where

$$\begin{aligned} Y_i^n(t) &= \tilde{A}_i^n(t \wedge \tau_n), \\ \tilde{Y}_i^n(t) &= x_i - \tilde{X}_i^n(0) - (\tilde{X}_i^n)^-(t) + (y_i - y_i^n)t + \tilde{S}_{\mu_i}^n\left(\frac{1}{N^n}T_i^n(t \wedge \tau_n)\right) - (1 - \mu_i\theta_i^n)Z_i^n(t \wedge \tau_n), \\ \tilde{Z}_i^n(t) &= \mu_i\theta_i^n Z_i^n(t). \end{aligned}$$

Define $W^n(t) = x + yt + Y^n(t) - \tilde{Y}^n(t) + \hat{\alpha}_\theta[Y^n, \tilde{Y}^n](t)$.

By (3.29), we have $\sup_i \sup_{[0, \tau_n]} |\rho_i t - \frac{1}{N^n}T_i^n(t)| \rightarrow 0$ as $n \rightarrow \infty$. Therefore using the regularity property of α_θ (3.19) and (3.20), (3.23) with a proper choice of $r_1 < \varepsilon$ we have

$$\|\varphi - W_n\|_{\tau_n}^* \leq \kappa_4 \varepsilon. \quad (3.30)$$

for all n large and $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$ and some constant κ_4 . Denote by ω_h $[\omega_g]$ the modulus of continuity of h [resp., g] over $\{q : \|q\| \leq \|\varphi\|_T^* + \kappa_3 + \kappa_4\}$. Hence using minimality property of $\hat{\alpha}_\theta$ we have

$$\begin{cases} h(\tilde{X}^n(t)) & \geq h((\tilde{X}_t^n)^+) - \omega_h(\sqrt{I}(c+6)r_1) \geq h(W^n(t)) - \omega_h(\sqrt{I}(c+6)r_1), \\ g(\tilde{X}^n(t)) & \geq h(W^n(t)) - \omega_g(\sqrt{I}(c+6)r_1) \geq h(W^n(t)) - \omega_g(\sqrt{I}(c+6)r_1). \end{cases} \quad (3.31)$$

for $t \in [0, \tau_n]$ and $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$ for large n .

Then by (3.30) and (3.31), for $(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}_r$ and all large n ,

$$\begin{aligned} \int_0^{\tau_n} h(\tilde{X}_n(s))ds &\geq \int_0^{\tau_n} h(W_n(s))ds - T\omega_h(\sqrt{I}(c+6)\varepsilon) \\ &\geq \int_0^{\tau_n} h(\varphi(s))ds - T\omega_h(\kappa_4\varepsilon) - T\omega_h(\sqrt{I}(c+6)\varepsilon). \end{aligned}$$

Combined with (3.28) this gives

$$\int_0^T h(\tilde{X}_n(s))ds \geq \int_0^T h(\varphi(s))ds - T\left(\omega_h(\kappa_4\varepsilon) + \omega_h(\sqrt{I}(c+6)\varepsilon) + \delta(\varepsilon)\right).$$

A similar argument gives

$$g(\tilde{X}_n(T)) = g(\varphi(T))\chi_{\{T \leq \tau_n\}} + g(\varphi(T))\chi_{\{T > \tau_n\}} \geq g(\varphi(T)) - \left(\omega_g(\kappa_4\varepsilon) + \omega_g(\sqrt{I}(c+6)\varepsilon) + \delta(\varepsilon) \right).$$

Hence using condition 3.2, it is easy to show that for all n large

$$\begin{aligned} \mathbb{E}[e^{b_n^2 \int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))}] &\geq \mathbb{E}\left[e^{b_n^2 \int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))} \chi_{\{(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r\}}\right] \\ &\geq e^{b_n^2 \left(\int_0^T h(\varphi(s))ds + g(\varphi(T)) - \mathbb{I}(\tilde{\psi}) - a(\varepsilon) - \varepsilon \right)}, \end{aligned}$$

where $a(\varepsilon) = T\omega_h(\kappa_4\varepsilon) + T\omega_h(\sqrt{I}(c+6)\varepsilon) + \omega_g(\sqrt{I}(c+6)\varepsilon) + \omega_g(\kappa_4\varepsilon) + (T+1)\delta(\varepsilon)$. The proof follows letting $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$. \square

Theorem 3.7 *Assume Conditions 3.2 and 3.3 hold and $\lim \frac{N^n}{b_n \sqrt{n}} = 0$. Then $\liminf V_Q^n(\tilde{X}_n(0)) \geq V(x)$.*

Proof: We note that proof of Theorem 3.6 relies on two estimates, (3.28) and (3.30). To get these estimates, we note that for any $\kappa > 0$ and large n , $\theta^n \cdot \tilde{X}^n \geq \kappa + 1 \Rightarrow \theta \cdot \tilde{Q}^n \geq \kappa/2$ and $\|\tilde{X}^n - \tilde{Q}^n\|_T^* = o(1)$. Hence the proof follows. \square

3.2.2 Upper bound

Theorem 3.8 *Assume Conditions 3.2, 3.3 and 3.4 hold and $\lim \frac{N^n}{b_n \sqrt{n}} = 0$. Then $\limsup V_X^n(\tilde{X}^n(0)) \leq V(x)$.*

Remark 3.5 *If the functions h, g are bounded then Condition 3.4 is not required in the above statement.*

The proof is based on the construction of a suitable admissible policy. The main idea of the proof is similar to that appear in [1]. However, the proof appear here is complicated than that appear in [1]. The main difficulty we face here is due to the constrain on the policy that does not allow processor sharing. The idea is to make use of the preemptive behavior of the policy. We construct a policy that serves each class of customers over small time intervals (defined in suitable sense) and on average effort given to serve class i is $\approx \rho_i + \mathfrak{E}$ where the correction \mathfrak{E} is small and leads us to the correct limit.

Proof: Let $\Delta > 0$ be a given constant. Define

$$\mathcal{Q} = \{\psi \in D([0, T], \mathbb{R}^{2I}) : \mathbb{I}(\psi) \leq \Delta\}. \quad (3.32)$$

By the definition of the rate function \mathbb{I} (from Remark 3.3), \mathcal{Q} is a compact set containing absolutely continuous paths starting from zero (particularly, $\mathcal{Q} \subset P$), with derivative having L^2 -norm uniformly bounded. Consequently, there exists a constant $M = M_\Delta$ such that $\|\psi^1\|^* + \|\psi^2\|^* \leq M$ for all $\psi \in \mathcal{Q}$. Consider the set $D(M+1)$ (3.16), let $\varepsilon \in (0, 1)$ be given, and choose $\delta, \eta > 0, \delta \in (0, \varepsilon)$, as in (3.18) and (3.19), corresponding to ε and $\kappa = M+1$. It follows from

the L^2 bound alluded to above, that for each fixed Δ , the members of \mathcal{Q} are equicontinuous. Hence one can choose $v_0 \in (0, \eta)$ (depending on Δ), such that

$$\text{osc}_{v_0}(\psi_i^l) < \frac{\delta}{4\sqrt{2\mathbf{I}}}, \text{ for all } \psi = (\psi^1, \psi^2) \in \mathcal{Q}, l = 1, 2, i \in \mathcal{I}. \quad (3.33)$$

Recall

$$\mathcal{A}_r(\tilde{\psi}) = \{\psi \in D([0, T], \mathbb{R}^{2I}) : \mathbf{d}(\psi, \tilde{\psi}) < r\}.$$

Noting that, for any $f \in \mathcal{Y}$ (see Notations),

$$\begin{aligned} \|\psi(t) - \tilde{\psi}(t)\| &\leq \|\psi(t) - \tilde{\psi}(f(t))\| + \|\tilde{\psi}(f(t)) - \tilde{\psi}(t)\|, \\ |f(t) - t|_T^* &\leq T(e^{\|f\|^\circ} - 1), \end{aligned}$$

it follows, by the equicontinuity of the members of \mathcal{Q} , that it is possible to choose $v_1 > 0$ such that, for any $\tilde{\psi} \in \mathcal{Q}$,

$$\psi \in \mathcal{A}_{v_1}(\tilde{\psi}) \quad \text{implies} \quad \|\psi - \tilde{\psi}\|^* < \frac{\delta}{4}. \quad (3.34)$$

Let $v_2 = \min\{v_0, v_1, \frac{\varepsilon}{2}\}$. Since \mathcal{Q} is compact and \mathbb{I} is lower semicontinuous, one can find a finite number of members $\bar{\psi}^1, \bar{\psi}^2, \dots, \bar{\psi}^N$ of \mathcal{Q} , and positive constants v^1, \dots, v^N with $v^k < v_2$, satisfying $\mathcal{Q} \subset \cup_k \mathcal{A}^k$, and

$$\inf\{\mathbb{I}(\psi) : \psi \in \overline{\mathcal{A}^k}\} \geq \mathbb{I}(\bar{\psi}^k) - \frac{\varepsilon}{2}, \quad k = 1, 2, \dots, N, \quad (3.35)$$

where, throughout, $\mathcal{A}^k := \mathcal{A}_{v^k}(\bar{\psi}^k)$.

Next we define a suitable policy such that the lower bound is asymptotically attained. Let $v = \frac{v_2}{2} \wedge \frac{T}{4}$ and $L = L(v) = \lfloor \frac{T}{v} \rfloor$. Define $a^\ell = \ell \cdot \frac{T}{L+1}$. Then $[a^\ell, a^{\ell+1})$ forms a disjoint partition of $[0, T)$ with $a^0 = 0$ and $\sup_i |a^{\ell+1} - a^\ell| < v$. Now consider a sequence $\{\alpha_n\}$ such that $\frac{\alpha_n \sqrt{n}}{b_n} \rightarrow 0$ as $n \rightarrow \infty$. Define $H^n = \lfloor \frac{\tilde{v}}{\alpha_n} \rfloor$ where $\tilde{v} = \frac{T}{L+1}$. We can choose $\{\alpha_n\}$ small enough so that $\inf_n H^n \geq 2$. Define

$$b_n^{\ell j} = a^\ell + j \frac{\tilde{v}}{H^n + 1}, \quad j = 1, \dots, H^n + 1.$$

Hence $\{[b_n^{\ell j}, b_n^{i(j+1)})\}_{j=0}^{j=H^n}$ forms a disjoint partition of $[a^\ell, a^{\ell+1})$ where $b_n^{\ell 0} = a^\ell, b_n^{\ell(H^n+1)} = a^{\ell+1}$. Also $\delta_n := \frac{\tilde{v}}{H^n+1} < \alpha_n$ for all n . Now we will split each $[b_n^{\ell j}, b_n^{\ell(j+1)})$, $\ell = 1, 2, \dots, L+1, j = 0, \dots, H^n$, using some random intervals. For this we define

$$\begin{cases} T_i^n = \int_0^\cdot B_i^n(s) ds, \\ \tilde{D}_i^n = \tilde{S}_{\mu_i}^n \circ (\frac{1}{N^n} T_i^n), \\ P_n = (\tilde{A}^n, \tilde{D}^n). \end{cases} \quad (3.36)$$

We also denote $\tilde{\ell} = f(x \cdot \theta) - x$ and

$$F_i^n(a^\ell) = \frac{b_n}{\mu_i \sqrt{n}} \frac{\hat{\alpha}_\theta[P_n]_i(a^{\ell-1}) - \hat{\alpha}_\theta[P_n]_i(a^{\ell-2})}{\tilde{v}}, \quad \ell \geq 2. \quad (3.37)$$

$\theta \cdot \hat{\alpha}_\theta$ being nondecreasing we have $\sum_i F_i^n(a^\ell) \geq 0$ for all $\ell \geq 2$. We need to define some more variable before we define the policy. We define random variables $\beta_i^n(\ell), \ell \geq 2$, as follows:

$$\beta_i^n(\ell) = \begin{cases} (\rho_i - F_i^n(a^\ell))^+ & \text{if } \sum_{i \in \mathcal{I}} (\rho_i - F_i^n(a^\ell))^+ \leq 1 \text{ and } \|P_n\|_{a^{\ell-1}}^* < M + 2, \\ \rho_i & \text{otherwise.} \end{cases}$$

We also define for $i \in \mathcal{I}$,

$$\gamma_i^n = \begin{cases} (\rho_i - \frac{b_n}{\mu^n \sqrt{n}} \frac{\tilde{\ell}_i}{v})^+ & \text{if } \sum_{i \in \mathcal{I}} (\rho_i - \frac{b_n}{\mu^n \sqrt{n}} \frac{\tilde{\ell}_i}{v})^+ \leq 1 \\ \rho_i & \text{otherwise.} \end{cases}$$

Now we split the intervals $[b_n^{ij}, b_n^{i(j+1)}), \ell = 0, 1, \dots, L, j = 0, 1, \dots, H^n$, using the above variables as follows: For $\ell = 0, j = 0, \dots, H^n$, we define

$$c_n^{0j}(i) = b_n^{0j} + \delta_n \sum_{k=1}^i \gamma_k^n, \quad i = 1, 2, \dots, \mathbf{I}.$$

We fix the notation as $c_n^{\ell j}(0) = b_n^{\ell j}$ and $c_n^{\ell j}(\mathbf{I} + 1) = b_n^{\ell(j+1)}$ for all ℓ, j . For $\ell = 1, j = 0, \dots, H^n$, we define

$$c_n^{1j}(i) = b_n^{1j} + \delta_n \sum_{k=1}^i \rho_k, \quad i = 1, 2, \dots, \mathbf{I}.$$

Finally, for $\ell \geq 2, j = 0, \dots, H^n$, we define

$$c_n^{\ell j}(i) = b_n^{1j} + \delta_n \sum_{k=1}^i \beta_k^n(\ell), \quad i = 1, 2, \dots, \mathbf{I}.$$

It is easy to see that $\{[c_n^{\ell j}(i), c_n^{\ell j}(i+1))\}_{i=0}^{\mathbf{I}}$ forms a partition of $[b_n^{\ell j}, b_n^{\ell(j+1)})$ for all $\ell = 1, \dots, L + 1, j = 1, \dots, H^n + 1$. Now we are ready to define the policy. Recall from (3.3), (3.5) that

$$\begin{cases} D_i^n = S_i \circ (\mu_i^n T_i^n), \\ X_i^n = X_i^n(0) + A_i^n - D_i^n. \end{cases} \quad (3.38)$$

For $i \in \mathcal{I}$, assume B_i^n is given by

$$B_i^n(t) = C_i^n(t) \wedge X_i^n(t), \quad t \in [0, T], \quad (3.39)$$

where

$$C_i^n(t) = \begin{cases} N^n & \text{if } t \in [c_n^{\ell j}(i-1), c_n^{\ell j}(i)) \text{ for some } \ell, j \\ 0 & \text{otherwise.} \end{cases} \quad (3.40)$$

Now let us argue that B^n is indeed an admissible policy. First we note that X^n, A^n, D^n are piecewise constant processes. Since γ^n is deterministic, the policy is well defined on $[0, a^2)$ with RCLL paths. This can be seen by applying induction on the jump times. Since $\beta_i^n(\ell), \ell \geq 2$, is completely determined by the values of P_n on $[0, a^{\ell-1}]$, B^n is uniquely defined on $[0, T)$. Fix

$B_i^n(T) = 0$ for all $i \in \mathcal{I}$. It is easy to check that B^n satisfies all the requirement for being admissible control. Hence $B^n \in \mathfrak{U}^n$ for all n . Hence by definition,

$$V_X^n(\tilde{X}^n(0)) \leq J_X^n(\tilde{X}^n(0), B^n). \quad (3.41)$$

With the policy defined above, we prove the result for upper bound. In what follows c_1, c_2, \dots , denote constants independent of $\Delta, \varepsilon, \delta, \eta, v$ and n .

Define $\varphi^k(t) = f(\varphi_\theta[\bar{\psi}^k](t))$ where $\bar{\psi}^k = x + yt + \psi^{k,1}(t) - R[\psi^{k,2}](t)$. Recall from (3.14) that φ^k is the dynamics corresponding to $\bar{\psi}^k$ and $\alpha_\theta[\bar{\psi}^k]$. Let $\tilde{A}_n = \Lambda_T(\tilde{A}^n) + \Lambda_T(\tilde{S}_\mu^n)$ and denote by Ω_k^n the event $\{(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}^k\}$. We prove the result in number of steps. First we show that for a constant c_1 , for all $n \geq n_0(\varepsilon, v)$,

$$\|\tilde{X}^n\|_T^* \leq c_1(1 + \tilde{A}_n), \quad (3.42)$$

and

$$\sup_{[\tilde{v}, T]} \|\tilde{X}^n - \varphi^k\| \leq c_1 \varepsilon, \quad \text{on } \Omega_k^n, k = 1, 2, \dots, N. \quad (3.43)$$

Step 1: From (3.17) and the fact $\frac{1}{N^n} T_i^n(t) \leq t$, we see that there exists c_2 such that

$$\sup_{a^\ell \in [0, t]} \|F^n(a^\ell)\| \leq \frac{b_n}{\sqrt{n}} \frac{c_2}{\tilde{v}} (1 + \|P_n\|_t^*). \quad (3.44)$$

Since $\rho_i \in (0, 1)$ for all $i \in \mathcal{I}$, we note from (3.44) that for all sufficiently large n , for any $\ell \geq 2$,

$$\|P_n\|_{a^{\ell-1}}^* < M + 2 \quad \text{implies} \quad \sum_i (\rho_i - F_n^i(a^\ell))^+ = \sum_i (\rho_i - F_n^i(a^\ell)) \leq 1,$$

as $\sum_i F_n^i(a^\ell) \geq 0$ for all $\ell \geq 2$. Define

$$\hat{\tau}_n = \min\{\ell \geq 1 : \|P_n\|_{a^\ell}^* \geq M + 2\}.$$

First we consider the event $\{\hat{\tau}_n = 1\}$. By definition, $\beta_i^n(\ell) = \rho_i$ for all $\ell \geq 2$ on $\{\hat{\tau}_n = 1\}$. For all large n , given $t \in [b_n^{\ell j}, b_n^{\ell(j+1)})$, for some $\ell = 0, 1, \dots, L$, $j = 0, 1, \dots, H^n$, we have on $\{\hat{\tau}_n = 1\}$,

$$\begin{aligned} & |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \\ & \leq |\rho_i a^\ell - \frac{1}{N^n} \int_0^{a^\ell} C_i^n(s) ds| + |\rho_i (b_n^{\ell j} - a^\ell) - \frac{1}{N^n} \int_{a^\ell}^{b_n^{\ell j}} C_i^n(s) ds| \\ & \quad + |\rho_i (t - b_n^{\ell j}) - \frac{1}{N^n} \int_{b_n^{\ell j}}^t C_i^n(s) ds|. \end{aligned} \quad (3.45)$$

If $\ell = 0$, then the first term on the r.h.s. disappears and the second term is equal to

$$|\rho_i j \delta_n - \frac{1}{N^n} j \delta_n \gamma_i^n N^n| \leq j \delta_n \left| \frac{b_n}{\mu^i \sqrt{n}} \frac{\tilde{\ell}_i}{\tilde{v}} \right| \leq \left| \frac{b_n}{\mu^i \sqrt{n}} \tilde{\ell}_i \right|.$$

If $\ell \geq 1$, on $\{\hat{\tau}_n = 1\}$, second term on the r.h.s. of (3.45) is equal to 0 and the first term less than equal to $\left| \frac{b_n}{\mu^i \sqrt{n}} \tilde{\ell}_i \right|$. Hence using the fact that $C_i^n \leq N^n$ and $|t - b_n^{\ell j}| \leq \delta_n$ we have

$$\sup_{t \in [0, T]} \frac{\sqrt{n}}{b_n} |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \leq c_3,$$

on $\{\hat{\tau}_n = 1\}$ for all n large. Now we consider the event $\{\hat{\tau}_n > 1\}$. For $t \in [b_n^{\ell j}, b_n^{\ell(j+1)}), t < a^{\hat{\tau}_n+1}$, for some $\ell = 0, 1, \dots, L, j = 0, 1, \dots, H^n$, we have

$$\begin{aligned} & |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \\ & \leq |\rho_i a^\ell - \frac{1}{N^n} \int_0^{a^\ell} C_i^n(s) ds| + |\rho_i (b_n^{\ell j} - a^\ell) - \frac{1}{N^n} \int_{a^\ell}^{b_n^{\ell j}} C_i^n(s) ds| \\ & \quad + |\rho_i (t - b_n^{\ell j}) - \frac{1}{N^n} \int_{b_n^{\ell j}}^t C_i^n(s) ds|. \end{aligned} \quad (3.46)$$

Now if $\ell \leq 1$, then a similar argument as above holds to bound the r.h.s. of (3.46). So we consider $\ell \geq 2$. Then for all n large

$$\begin{aligned} & |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \\ & = |\rho_i t - \frac{1}{N^n} \left(\int_0^{a^1} C_i^n(s) ds + \int_{a^1}^{a^\ell} C_i^n(s) ds + \int_{a^\ell}^{b_n^{\ell j}} C_i^n(s) ds + \int_{b_n^{\ell j}}^t C_i^n(s) ds \right)| \\ & \leq \frac{b_n}{\sqrt{n}} c_3 + |\rho_i (a^\ell - a^2) - \frac{1}{N^n} N^n \tilde{v} \sum_{k=2}^{\ell-1} \beta_i^n(k)| \\ & \quad + |\rho_i (b_n^{\ell j} - a^\ell) - \frac{1}{N^n} N^n j \delta_n \beta_i^n(k)| + 2\delta_n. \\ & \leq \frac{b_n}{\sqrt{n}} c_3 + |\tilde{v} \sum_{k=2}^{\ell-1} F_i^n(a^k)| + |j \delta_n F_i^n(a^\ell)| + 2\delta_n \\ & \leq \frac{b_n}{\sqrt{n}} c_3 + \left| \frac{b_n}{\mu_i \sqrt{n}} (\hat{\alpha}_\theta[P_n](a^{\ell-2}) - \hat{\alpha}_\theta[P_n](0)) \right| + |j \delta_n F_i^n(a^\ell)| + 2\delta_n \\ & \leq \frac{b_n}{\sqrt{n}} c_4 (1 + \|P_n\|_t^*) + 2\delta_n. \end{aligned}$$

So now we are left with case $t \in [b_n^{\ell j}, b_n^{\ell(j+1)}), t \geq a^{\hat{\tau}_n+1}$, for some $\ell = 0, 1, \dots, L, j = 0, 1, \dots, H^n$, on the event $\{\hat{\tau}_n > 1\}$. We note that,

$$\begin{aligned} & |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \\ & \leq |\rho_i a^{\hat{\tau}_n+1} - \frac{1}{N^n} \int_0^{a^{\hat{\tau}_n+1}} C_i^n(s) ds| + |\rho_i (t - a^{\hat{\tau}_n+1}) - \frac{1}{N^n} \int_{a^{\hat{\tau}_n+1}}^t C_i^n(s) ds| \\ & \leq \frac{b_n}{\sqrt{n}} c_4 (1 + \|P_n\|_t^*) + 4\delta_n. \end{aligned}$$

Hence combining all the calculations above and making use of the fact that $\frac{\delta_n \sqrt{n}}{b_n} \rightarrow 0$ as $n \rightarrow \infty$ we have a constant c_5 such that

$$\sup_{t \in [0, T]} \frac{\sqrt{n}}{b_n} |\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \leq c_5 (1 + \tilde{A}_n), \quad (3.47)$$

for all n large.

Step 2: Now we are ready to prove (3.42). Rewrite (3.8) as $(\tilde{X}_i^n - \varepsilon)^+ = \hat{Y}_i^n + \hat{Z}_i^n$, where

$$\begin{aligned}\hat{Y}_i^n(t) &= (\tilde{X}_i^n - \varepsilon)^- + \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n\left(\frac{1}{N^n} T_n^i(t)\right) \\ &\quad + \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds\right) \\ \hat{Z}_i^n(t) &= \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \frac{1}{N^n} \int_0^t (C_i^n(s) - X_i^n(s))^+ ds.\end{aligned}$$

Now for each $i \in \mathcal{I}$, $(\tilde{X}_i^n - \varepsilon)^+$ is nonnegative and \hat{Z}_i^n is nonnegative, nondecreasing. Since $\frac{N^n}{b_n \sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$, for all n large (depending on ε),

$$(\tilde{X}_i^n(s) - \varepsilon)^+ > 0 \Rightarrow X_i^n(s) > \rho_i N^n + \varepsilon \sqrt{n} b_n \Rightarrow X_i^n(s) > N^n \Rightarrow (C_i^n(s) - X_i^n(s))^+ = 0.$$

Therefore $\int_0^\cdot (\tilde{X}_i^n(s) - \varepsilon)^+ d\hat{Z}_i^n(s) = 0$. Therefore $((\tilde{X}_i^n - \varepsilon)^+, \hat{Z}_i^n)$ is the solution to the Skorohod problem for data \hat{Y}_i^n . Hence applying Lipschitz property of the Skorohod map and (3.47), we have

$$|\hat{Z}_i^n|_T^* + |(\tilde{X}_i^n - \varepsilon)^+|_T^* \leq 4|\hat{Y}_i^n|_T^* \leq c_6(1 + \tilde{A}_n), \quad (3.48)$$

for all n large where we used the fact that $(\tilde{X}_i^n - \varepsilon)^- \leq \varepsilon + \frac{\rho_i N^n}{b_n \sqrt{n}} < 1$. Now (3.42) follows from (3.48). Since

$$\frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t B_i^n(s) ds\right) = \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds\right) + \hat{Z}_i^n,$$

using (3.47), (3.48) and convergence of $\frac{\mu_i^n N^n}{n}$, we have

$$\sup_{t \in [0, T]} \left| \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \left| \rho_i t - \frac{1}{N^n} T_i^n(t) \right| \right| \leq c_7(1 + \tilde{A}_n), \quad (3.49)$$

for all n large.

Step 3: In particular, (3.49) implies that for all n large,

$$\sup_{t \in [0, T]} \left| \rho_i t - \frac{1}{N^n} T_i^n(t) \right| \leq \frac{\tilde{v}}{2} \quad (3.50)$$

holds on the event $\cup_k \Omega_k^n$. Therefore using (3.34), (3.33) and (3.49) one obtains that for all large n (see (66) in [1]),

$$\sup_{t \in [\tilde{v}, T]} \|\tilde{S}_\mu^n(T_n(t)) - R[\bar{\psi}^{k,2}](t - \tilde{v})\| \leq \frac{\delta}{2}, \quad (3.51)$$

on $\Omega_k^n, k = 1, 2, \dots, N$. In the rest of this step, we calculate the difference between $Z^n(t)$ (see (3.9)) and $\alpha_\theta[\bar{\psi}](t - \tilde{v})$ on the event Ω_k^n . Recall $\hat{\tau}_n$ defined above. We note that on Ω_k^n one has $\hat{\tau}_n > L$ for all n large since $\|P_n\|_T^* \leq \|\tilde{A}^n\|^* + \|\tilde{S}_\mu^n\|^* < M + 2$ by (3.34). Hence for all n large, on $\Omega_k^n, k = 1, 2, \dots, N$,

$$\gamma_i^n = \left(\rho_i - \frac{b_n}{\mu_i^n \sqrt{n}} \frac{\tilde{\ell}_i}{\tilde{v}}\right) \quad \text{and} \quad \beta_i^n(\ell) = (\rho_i - F_i^n(a^\ell)),$$

for $\ell = 2, 3, \dots, L$. Therefore for any $t \in [b^{0j}, b^{0(j+1)})$, $j = 1, \dots, H^n$, we have

$$\begin{aligned}
|\mu_i \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds) - \frac{t}{\tilde{v}} \tilde{\ell}_i| &= |\mu_i \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} \int_0^{b^{0j}} C_i^n(s) ds - \frac{1}{N^n} \int_{b^{0j}}^t C_i^n(s) ds) - \frac{t}{\tilde{v}} \tilde{\ell}_i| \\
&\leq \mu_i \frac{\sqrt{n}}{b_n} (t - j\delta_n) + \frac{t - j\delta_n}{\tilde{v}} \tilde{\ell}_i + \mu_i \frac{\sqrt{n}}{b_n} \delta_n \\
&\leq 2\mu_i \frac{\sqrt{n}}{b_n} \alpha_n + \frac{1}{H^n + 1} \tilde{\ell}_i.
\end{aligned} \tag{3.52}$$

Now for $k = 1, 2, \dots, N$, consider

$$\hat{W}_{i,k}^n(t) := \mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds \right) - \alpha_\theta^i[\bar{\psi}^k]_i(t - \tilde{v}), \quad t \in [\tilde{v}, T],$$

on the event Ω_n^k . We note from (3.14) that $\alpha_\theta[\bar{\psi}^k](0) = \tilde{\ell}$. Hence for $t \in [a^1, a^2)$ (recall $a^\ell = \ell\tilde{v}$) and all large n , we have from (3.33) and (3.19) that

$$\begin{aligned}
|\hat{W}_{i,k}^n(t)| &\leq |W_{i,k}^n(\tilde{v}) - \tilde{\ell}_i| + |\mu_i \frac{\sqrt{n}}{b_n} (\rho_i(t - a^1) - \frac{1}{N^n} \int_{a^1}^t C_i^n(s) ds)| + |\tilde{\ell}_i - \alpha_\theta^i[\bar{\psi}^k]_i(t - v)| \\
&\leq 4\mu_i \frac{\sqrt{n}}{b_n} \alpha_n + \frac{1}{H^n + 1} \tilde{\ell}_i + \varepsilon,
\end{aligned} \tag{3.53}$$

where we use (3.52) to estimate the first term and a similar estimate to calculate the second (for example, put $\tilde{\ell}_i = 0$ in (3.52)). Now we consider $t \in [a^2, T)$. Let $t \in [b_n^{\ell j}, b_n^{\ell(j+1)})$ for some $\ell \geq 2, j = 0, \dots, H^n$. The following calculations are of same type as step 1. We note that for large n , on Ω_k^n ,

$$\begin{aligned}
&\mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i b_n^{\ell j} - \frac{1}{N^n} \int_0^{b_n^{\ell j}} C_i^n(s) ds \right) \\
&= \mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i a^\ell - \frac{1}{N^n} \int_0^{a^\ell} C_i^n(s) ds \right) + \mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i (a^\ell - b_n^{\ell j}) - \frac{1}{N^n} \int_{a^\ell}^{b_n^{\ell j}} C_i^n(s) ds \right) \\
&= \mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i a^\ell - \frac{1}{N^n} N^n \tilde{v} (\gamma_i^n + \rho_i + \sum_{k=2}^{\ell-1} \beta_i^n(k)) \right) + \mu_i \frac{\sqrt{n}}{b_n} \left(\rho_i (a^\ell - b_n^{\ell j}) - \frac{1}{N^n} N^n j \delta_n \beta_i^n(\ell) \right) \\
&= \hat{\alpha}_\theta[P_n]_i((\ell - 2)\tilde{v}) + \frac{j\delta_n}{\tilde{v}} [\hat{\alpha}_\theta[P_n]_i((\ell - 1)\tilde{v}) - \hat{\alpha}_\theta[P_n]_i((\ell - 2)\tilde{v})].
\end{aligned}$$

Therefore using (3.18), (3.19), (3.33), (3.34) and (3.50), we see that for all n large, on $\Omega_k^n, k = 1, 2, \dots, N$,

$$\begin{aligned}
|\hat{\alpha}_\theta[P_n]_i((\ell - 2)\tilde{v}) - \alpha_\theta[\bar{\psi}^k]_i(t - \tilde{v})| &\leq c_8 \varepsilon \\
|\hat{\alpha}_\theta[P_n]_i((\ell - 1)\tilde{v}) - \hat{\alpha}_\theta[P_n]_i((\ell - 2)\tilde{v})| &\leq c_8 \varepsilon \\
|\frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds \right) - \frac{\sqrt{n}}{b_n} \left(\rho_i b_n^{\ell j} - \frac{1}{N^n} \int_0^{b_n^{\ell j}} C_i^n(s) ds \right)| &\leq 2 \frac{\sqrt{n}}{b_n} \alpha_n,
\end{aligned}$$

for some constant c_8 independent of t (see (68) in [1]). Hence combining all these calculations with (3.53) and using the fact that $\frac{\sqrt{n}}{b_n}\alpha_n \rightarrow 0$, we have, for all n large and all k ,

$$\sup_{t \in [\tilde{v}, T]} \left| \frac{N^n \mu_i^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds \right) - \alpha_\theta[\bar{\psi}^k]_i(t - \tilde{v}) \right| \leq c_9 \varepsilon, \quad (3.54)$$

on Ω_n^k .

Step 4: Recall $\varphi^k(t) = f(\varphi_\theta[\bar{\psi}^k](t))$. The goal of this step is to estimate the difference between \tilde{X}_n and φ^k on Ω_n^k . To this end, let first

$$\tilde{\varphi}^k(t) = \begin{cases} x + \frac{t}{\tilde{v}} \ell & \text{for } t \in [0, \tilde{v}) \\ f(\varphi_\theta[\bar{\psi}^k](t - \tilde{v})) & \text{for } t \in [\tilde{v}, T]. \end{cases}$$

Also recall from step 2 that $(\tilde{X}_i^n - \varepsilon)^+$ solves Skorohod problem for the date \hat{Y}_i^n . Since $\Gamma(\tilde{\varphi}^k) = \tilde{\varphi}^k$ for all $k = 1, \dots, N$, we have for large n ,

$$|(\tilde{X}_i^n - \varepsilon)^+ - \tilde{\varphi}_i^k|_T^* \leq 2|\hat{Y}_i^n - \tilde{\varphi}_i^k|_T^*. \quad (3.55)$$

Now for $t \in [0, \tilde{v})$,

$$\begin{aligned} & |\hat{Y}_i^n(t) - \tilde{\varphi}_i^k(t)| \\ & \leq |(\tilde{X}_i^n - \varepsilon)^- + \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n(\frac{1}{N^n} T_n^i(t)) \\ & \quad + \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds) - x_i - \frac{t}{\tilde{v}} \tilde{\ell}_i| \\ & \leq c_{10} \varepsilon, \end{aligned}$$

for all n large where we use (3.52), (3.33) and (3.34). Similarly, using (3.51), (3.54), for $t \in [\tilde{v}, T]$,

$$\begin{aligned} & |\hat{Y}_i^n(t) - \tilde{\varphi}_i^k(t)| \\ & \leq |(\tilde{X}_i^n - \varepsilon)^- + \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n(\frac{1}{N^n} T_n^i(t)) - \bar{\psi}_i^{k,1}(t - \tilde{v}) + R[\bar{\psi}^{k,2}]_i(t - \tilde{v}) \\ & \quad + \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} (\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds) - x_i - y_i(t - \tilde{v}) - \alpha_\theta[\bar{\psi}^k]_i(t - \tilde{v})| \\ & \leq c_{11} \varepsilon. \end{aligned}$$

Therefore from (3.55), we get that for large n , on $\Omega_n^k, k = 1, \dots, N$,

$$|\tilde{X}_i^n - \tilde{\varphi}_i^k|_T^* \leq c_{12} \varepsilon.$$

Thus (3.43) follows by comparing $\tilde{\varphi}^k$ and φ^k .

Rest of the proof follows by standard argument using (3.42) and (3.43) (see for example, Step 5 in [1]). \square

If $\frac{N^n}{b_n \sqrt{n}} \rightarrow 0$, then $\|\tilde{X}^n - \tilde{Q}^n\|_T^* \rightarrow 0$ as $n \rightarrow \infty$. Hence it is easy to obtain estimates like (3.42) and (3.43) for \tilde{Q}^n with the policy constructed in Theorem 3.8 when $\frac{N^n}{b_n \sqrt{n}} \rightarrow 0$. Thus we have the following theorem:

Theorem 3.9 *Let Conditions 3.2, 3.3 and 3.4 hold and $\lim_{n \rightarrow \infty} \frac{N^n}{b_n \sqrt{n}} = 0$. Then*

$$\limsup_{n \rightarrow \infty} V_Q^n(\tilde{Q}^n(0)) \leq V(x).$$

Theorem 3.10 *Let Conditions 3.2 and 3.3 hold. If h and g are bounded, then*

$$\limsup_{n \rightarrow \infty} V_X^n(\tilde{X}^n(0)) \leq V(x).$$

Proof: From Theorem 3.8, we see that we only need to consider the case when $\limsup \frac{N^n}{b_n \sqrt{n}} > 0$.

Instead of introducing a new subsequence, we assume that $\lim \frac{N^n}{b_n \sqrt{n}} > 0$. Hence $\lim \frac{\sqrt{n}}{b_n N^n} = 0$.

Given $\varepsilon \in (0, 1)$, we construct an ε -optimal policy. Since h, g are bounded, it is enough to construct a policy so that (3.43) holds for large n .

Let $\Delta > 0$ be given. Define

$$\mathcal{Q} = \{\psi \in D([0, T], \mathbb{R}^{2\mathbf{I}}) : \mathbb{I}(\psi) \leq \Delta\}. \quad (3.56)$$

Hence $\mathcal{Q} \subset D(M)$ for a suitably chosen M . Using the same argument as in Theorem 3.8, we have $v \leq \varepsilon/2$ such that

$$\text{osc}_v(\psi_i^l) < \frac{\delta}{4\sqrt{2\mathbf{I}}}, \text{ for all } \psi = (\psi^1, \psi^2) \in \mathcal{Q}, l = 1, 2, i \in \mathcal{I}, \quad (3.57)$$

where $\delta \in (0, \varepsilon)$ is chosen according to (3.18) and

$$\psi \in \mathcal{A}_v(\tilde{\psi}) \quad \text{implies} \quad \|\psi - \tilde{\psi}\|^* < \frac{\delta}{4}, \quad (3.58)$$

for all $\tilde{\psi} \in \mathcal{Q}$. Also we can find finite number of members $\bar{\psi}^1, \bar{\psi}^2, \dots, \bar{\psi}^N$ of \mathcal{Q} , and positive constants v^1, \dots, v^N with $v^k < v$, satisfying $\mathcal{Q} \subset \cup_k \mathcal{A}^k$, and

$$\inf\{\mathbb{I}(\psi) : \psi \in \overline{\mathcal{A}^k}\} \geq \mathbb{I}(\bar{\psi}^k) - \frac{\varepsilon}{2}, \quad k = 1, 2, \dots, N, \quad (3.59)$$

where, throughout, $\mathcal{A}^k := \mathcal{A}_{v^k}(\bar{\psi}^k)$. Define

$$F_i^n(t) = \frac{b_n}{\mu_i \sqrt{n}} \frac{\hat{\alpha}_\theta[P_n]_i(jv) - \hat{\alpha}_\theta[P_n]_i((j-1)v)}{v}, \quad \text{for } jv \leq t < (j+1)v, j \geq 1, \quad (3.60)$$

where $P_n = (\tilde{A}^n, \tilde{D}^n)$ (3.36). Since $\hat{\alpha}_\theta$ satisfies the causality property, F^n is well defined. Denote

$$\Theta(a, b) = a\chi_{\mathbb{R}^+}(a)\chi_{[0,1]}(b), \quad a, b \in \mathbb{R}.$$

Recall that $\tilde{\ell} = f(x \cdot \theta) - x$. Define

$$B_i^n(t) = \begin{cases} \Theta(\lfloor (\rho_i - \frac{b_n}{\mu_i \sqrt{n}} \frac{\tilde{\ell}_i}{v}) N^n \rfloor, \sum_{i \in \mathcal{I}} (\rho_i - \frac{b_n}{\mu_i \sqrt{n}} \frac{\tilde{\ell}_i}{v})^+) \chi_{\{\tilde{X}_i^n(t) > \varepsilon\}} \wedge X_i^n(t) & \text{if } t \in [0, v) \\ \lfloor \rho_i N^n \rfloor \chi_{\{\tilde{X}_i^n(t) > \varepsilon\}} \wedge X_i^n(t) & \text{if } t \in [v, 2v) \\ \Theta(\lfloor (\rho_i - F_i^n(t-v)) N^n \rfloor, \sum_{i \in \mathcal{I}} (\rho_i - F_i^n(t-v))^+) \chi_{\{\tilde{X}_i^n(t) > \varepsilon\}} \wedge X_i^n(t) & \text{otherwise.} \end{cases} \quad (3.61)$$

Using same argument as in Theorem 3.8, it is easy to see that B^n is an admissible control and hence $B^n \in \mathfrak{U}^n$. As earlier, define $\varphi^k(t) = f(\varphi_\theta[\bar{\psi}^k](t))$ where $\bar{\psi}^k = x + yt + \psi^{k,1}(t) - R[\psi^{k,2}](t)$. Denote by Ω_k^n the event $\{(\tilde{A}^n, \tilde{S}_\mu^n) \in \mathcal{A}^k\}$.

In what follows, c_1, c_2, \dots denote constants independent of $\Delta, \varepsilon, n, v, \delta, \eta$.

As earlier (proof of Theorem 3.8), it is enough to show that there exists a constant c_1 such for all $n \geq n_0(\varepsilon, v)$,

$$\sup_{[v, T]} \|\tilde{X}^n - \varphi^k\| \leq c_1 \varepsilon, \quad \text{on } \Omega_k^n, k = 1, 2, \dots, N. \quad (3.62)$$

First we note from (3.58) that $\|P_n\|_T^* < M + 2$ on Ω_k^n for $k = 1, 2, \dots, N$. Also from (3.17) and the fact $\frac{1}{N^n} T_i^n(t) \leq t$, we see that there exists d_2 such that

$$\sup_{s \in [v, t]} \|F^n(s)\| \leq \frac{b_n}{\sqrt{n}} \frac{c_2}{v} (1 + \|P_n\|_t^*). \quad (3.63)$$

Therefore for all n large, $\rho_i - \frac{b_n}{\mu_i \sqrt{n}} \frac{\tilde{\ell}_i}{v}, \rho_i - F_i^n(t - v), i \in \mathcal{I}$, are positive and

$$\sum_i (\rho_i - \frac{b_n}{\mu_i \sqrt{n}} \frac{\tilde{\ell}_i}{v}) \leq 1, \quad \sum_i \rho_i - F_i^n(t - v) \leq 1$$

on $\Omega_k^n, k = 1, 2, \dots, N$ where we use that $\theta \cdot \tilde{\ell} = 0$ and $\sum_i F_i^n \geq 0$. Again

$$\tilde{X}_i^n(t) > \varepsilon \Rightarrow X_i^n(t) > \rho_i N^n + \varepsilon b_n \sqrt{n} \Rightarrow \varepsilon \Rightarrow X_i^n(t) > (\rho_i + \varepsilon \frac{b_n}{\sqrt{n}} \frac{n}{N^n}) N^n.$$

Hence using (3.63) and the fact $\lim \frac{n}{N^n} = \infty$, we have on $\Omega_k^n, k = 1, 2, \dots, N$,

$$\tilde{X}_i^n(t) > \varepsilon \Rightarrow X_i^n(t) > (\rho_i - F_i^n(t - v)) N^n, \quad t \geq 2v,$$

for all n large. Similar fact holds in $[0, v)$. Therefore for large n , we have $B_i^n(t) = \lfloor C_i^n(t) \rfloor \chi_{\{\tilde{X}_i^n(t) > \varepsilon\}}$ on $\Omega_k^n, k = 1, 2, \dots, N$, where

$$C_i^n(t) = \begin{cases} (\rho_i - \frac{b_n}{\mu_i \sqrt{n}} \frac{\tilde{\ell}_i}{v}) N^n & \text{if } t \in [0, v) \\ \rho_i N^n & \text{if } t \in [v, 2v) \\ (\rho_i - F_i^n(t - v)) N^n & \text{otherwise.} \end{cases} \quad (3.64)$$

Using the same argument as (3.23), we have for large n ,

$$\sup_{i \in \mathcal{I}} |(\tilde{X}_i^n)^-|_T^* \leq (c + 6) \varepsilon, \quad (3.65)$$

on $\Omega_k^n, k = 1, 2, \dots, N$ where $c = \sup_n \|y^n\|$. Following the same arguments in (Step 1, [1]) we obtain, for large n ,

$$\sup_{[0, T]} \frac{\sqrt{n}}{b_n} |\rho_i - \frac{1}{N^n} \int_0^t C_i^n(s) ds| \leq c_4, \quad (3.66)$$

on Ω_k^n for some constant c_4 . We rewrite (3.8) as $(\tilde{X}_i^n - \varepsilon)^+ = \hat{Y}_i^n + \hat{Z}_i^n$ where

$$\begin{aligned}\hat{Y}_i^n(t) &= (\tilde{X}_i^n - \varepsilon)^- + \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_{\mu_i}^n\left(\frac{1}{N^n} T_n^i(t)\right) \\ &\quad + \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t C_i^n(s) ds\right) + \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \frac{1}{N^n} \int_0^t (C_i^n(s) - \lfloor C_i^n(s) \rfloor) ds \\ \hat{Z}_i^n(t) &= \frac{\mu_i^n N^n}{n} \frac{\sqrt{n}}{b_n} \frac{1}{N^n} \int_0^t \lfloor C_i^n(s) \rfloor \chi_{\{\tilde{X}_i^n(s) \leq \varepsilon\}} ds.\end{aligned}$$

Since $(\tilde{X}_i^n - \varepsilon)^+ > 0 \Rightarrow \chi_{\{\tilde{X}_i^n(s) \leq \varepsilon\}} = 0$, $(\tilde{X}_i^n - \varepsilon)^+$ solves Skorohod problem for the data \hat{Y}_i^n . Hence on Ω_k^n , for large n , $\sup_i |\hat{Z}_i^n|_T^* \leq c_5$ using the fact that

$$\sup_{[0, T]} \left| \frac{\sqrt{n}}{b_n} \frac{1}{N^n} \int_0^t (C_i^n(s) - \lfloor C_i^n(s) \rfloor) ds \right| \leq \frac{\sqrt{n}}{b_n} \frac{T}{N^n} \rightarrow 0.$$

Combining with (3.66), for large n ,

$$\sup_i \sup_{[0, T]} \left| \frac{\sqrt{n}}{b_n} \left| \rho_i - \frac{1}{N^n} \int_0^t B_i^n(s) ds \right| \right| \leq c_6, \quad (3.67)$$

on Ω_k^n for some constant c_6 . Now we can use the same arguments as in (Step 3, [1]) to conclude that for large n , on Ω_k^n ,

$$\sup_{[v, T]} \left| \frac{N^n \mu_i^n}{n} \frac{\sqrt{n}}{b_n} \left(\rho_i t - \frac{1}{N^n} \int_0^t B_i^n(s) ds \right) - \alpha_{\theta}^i [\bar{\psi}^k](t - v) \right| \leq c_7 \varepsilon, \quad (3.68)$$

for some constant c_7 . We define $\tilde{\varphi}^k$ as in Step 4 above replacing \tilde{v} by v . Using the Lipschitz property of the Skorohod map, we obtain on Ω_k^n , for large n ,

$$|(\tilde{X}_i^n)^+ - \tilde{\varphi}_i^k|_T^* \leq 2|\hat{Y}_i^n - \tilde{\varphi}_i^k|_T^* \leq c_8 \varepsilon,$$

where the last estimate is obtained using the same argument as Step 4 above. Combining with (3.65), we have on $\Omega_k^n, k = 1, 2, \dots, N$, $|\tilde{X}_i^n - \tilde{\varphi}_i^k|_T^* \leq c_9 \varepsilon$ for some constant c_9 and n large. Hence we obtain (3.62) comparing $\tilde{\varphi}$ and φ . \square

3.3 Linear cost and asymptotic optimality

In this section, we provide a simple policy based on priority that is asymptotically optimal. We assume that h and g have following forms

$$h(x) = \sum_{i=1}^{\mathbf{I}} c_i x_i, \quad g(x) = \sum_{i=1}^{\mathbf{I}} d_i x_i,$$

where c_i and d_i are nonnegative constants, and, in addition,

$$c_1 \mu_1 \geq c_2 \mu_2 \geq \dots \geq c_{\mathbf{I}} \mu_{\mathbf{I}} \quad \text{and} \quad d_1 \mu_1 \geq d_2 \mu_2 \geq \dots \geq d_{\mathbf{I}} \mu_{\mathbf{I}}.$$

We consider the $c\mu$ -rule that prioritizes according to the ordering of class labels, with highest priority to class 1. Define

$$B_1^n = X_1^n, B_2^n = X_2^n \wedge (N^n - B_1^n), \dots, B_I^n = X_I^n \wedge (N^n - \sum_{i < I} B_i^n). \quad (3.69)$$

It is easy to see that the above policy is consistent with (3.3)-(3.5) and $B^n \in \mathfrak{U}^n$. Proof of the following theorem follows using the same argument from Theorem 5.1 in [1].

Theorem 3.11 *Assume Conditions 3.2, 3.4 hold and $\lim_{n \rightarrow \infty} \frac{N^n}{b_n \sqrt{n}} = 0$. Then, under the priority policy $\{B^n\}$ of (3.69),*

$$\lim_{n \rightarrow \infty} J_Q^n(\tilde{Q}^n(0), B^n) = \lim_{n \rightarrow \infty} J_X^n(\tilde{X}^n(0), B^n) = V(x).$$

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